Part 2

"Quantum spin liquid" in the ground states of low dimensional AF quantum spin systems

Haldane gap and the VBS state for the $S=1$ AF chain

two $S=\frac{1}{2}$'s. $\uparrow_1 \quad \downarrow_2$

ferro $\hat{H} = -\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \xrightarrow{g.s.}$ triplet $\begin{cases} \uparrow\uparrow \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{cases}$

antiferro $\hat{H} = \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \xrightarrow{g.s.}$ singlet $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

"QUANTUF FLUTCTUATION"

Heisenberg AF

$d \geq 2$ the g.s. develops long-range Néel order

"quantum fluctuation" is $\begin{cases} \text{small} \uparrow \\ \text{large} \downarrow \end{cases}$

$d=1$ no long-range Néel order in the g.s.

$\downarrow$

?

⟨Haldane conjecture and related results⟩

$d=1$ (almost throughout the present part)

§ Haldane conjecture

Heisenberg AF chain $\hat{H} = \sum_{x=1}^{L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1}$ $(\hat{\mathbf{S}}_{L+1} = \hat{\mathbf{S}}_1)$

$(S = \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ (L even)

(Marshll-Lieb-Mattis theorem
→ the g.s. is unique for finite L.)

$\boxed{S=\frac{1}{2}}$ Common beliefs based on the Bethe ansatz solution (1931)

i) the g.s. is unique (also for $L \uparrow \infty$) → NO LRO or SSB

ii) no energy gap above the g.s. energy $\equiv$ $E_{1st} - E_{GS} = O(\frac{1}{L})$

iii) the g.s. correlation funct. decays by a power law as

$$\langle \Phi_{GS}, \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_{GS} \rangle \simeq (-1)^{x-y} |x-y|^{-1}$$

Haldan 1983

- non-linear σ-model with a topological term
- sem-classical quantization of solitons

large S limit

$S=\frac{1}{2}, \frac{3}{2}, \cdots$ half-odd-integer spins

i)
ii) as in $S=\frac{1}{2}$
iii)

$\boxed{\text{massless or critical}}$

$S=1,2,3,\cdots$ integer spins

i)' the g.s. is unique (also for $L\uparrow\infty$) → NO LRO or SSB

ii)' ∃ a nonvanishing energy gap above the g.s energy

Haldane gap ($\Delta E \simeq 2Se^{-\pi S}$) $\updownarrow O(1)$

iii)' the g.s. correlation function decays exponentially

$$\langle \Phi_{GS}, \hat{S}_x \cdot \hat{S}_y \Phi_{GS} \rangle \simeq (-1)^{x-y} \exp[-\frac{|x-y|}{\xi}]$$

massive or disordered

disordered (massive) behavior at $T=0$

strong "quantum fluctuation"

at least in mid 80's

Surprising points of the conjecture

- a drastic difference between the systems with half-odd-integer S and integer S
- it ~~is~~ seemed "natural" that a one-dim. system with a continuous symmetry has low-energy excitations.

see the next section

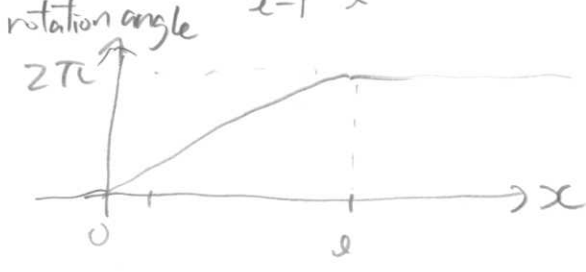
Rem

ii') ⇒ iii') was finally proved by Hastings and Koma 2006

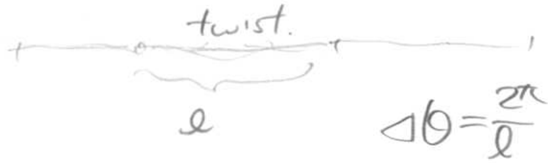
(the beginning of modern applications of the Lieb-Robinson bound)

§ Theorem which rules out "unique g.s. + gap"

twist operator $\hat{U}_l = \exp\left[i \sum_{x=1}^{l} \frac{2\pi x}{l} \hat{S}_x^{(3)} \right]$ $\quad l < L$



$\Psi = \hat{U}_l \Phi_{GS}$



$\Delta\theta = \frac{2\pi}{l}$

$\langle \Psi, \hat{H} \Psi \rangle - E_{GS} = l \cdot O((\Delta\theta)^2) = O(\frac{1}{l})$

always gapless ?!

One can prove $\langle \Phi_{GS}, \Psi \rangle = 0$ only for $S = \frac{1}{2}, \frac{3}{2}, \cdots$

Theorem (Lieb-Schultz-Mattis 1961, Affleck-Lieb 1986)

For $S = \frac{1}{2}, \frac{3}{2}, \cdots$ "unique g.s. + gap" is impossible.

No information for $S = 1, 2, \cdots$

generalization

(Yamanaka-Oshikawa-Affleck 1997)

DATE

§ Semi-classical approach

$$\hat{H} = \underbrace{\sum_{x=1}^{L} \hat{S}^{(3)}_x \hat{S}^{(3)}_{x+1}}_{\text{classical (Ising)} \;\to\; \hat{H}_c} + \underbrace{\frac{1}{2}\sum_{x=1}^{L} \left\{ \hat{S}^+_x \hat{S}^-_{x+1} + \hat{S}^-_x \hat{S}^+_{x+1} \right\}}_{\text{"quantum"} \;\to\; \hat{H}_q}$$

treat as "perturbation"

$S = \frac{1}{2}$ G.S. of $\hat{H}_c$

↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓

$\hat{S}^+\hat{S}^-$

↑ ↓ ↑ ↑ ↓ ↓ ↑ ↓

- pair creation of kinks

$\hat{S}^+\hat{S}^-$

↑ ↓ ↑ ↑ ↓ ↑ ↓ ↓

- kinks hop by twice the lattice spacing

also pair annihilation

Note that there are two kinds of kinks → even, odd

different kinds of kinks never pair-annihilate

↑ ↓ ↑ ↑ ↓ ↑ ↑ ↓ ↑

↑ ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↑

no way!!

$\underline{S=1}$ GS of $\hat{H}_c$ $+ - + - + - + -$

$\hat{S}^- \hat{S}^+$

$+ - 0\ 0 + - + -$ pair creation of kinks (0's)

$S^+ S^-$

$+ - 0 + 0 - + -$ kinks hop by a single lattice spacing

$S^0 S^+$

$+ - 0 + - 0 + -$

- only one kind of kinks, pairly created and annihilated.

$\Uparrow$

$\boxed{\text{essential difference from the } S=\tfrac{1}{2} \text{ case}}$ → massive behavior?

- this construction generates only special states like

$+ 0 - + - 0\ 0 + 0 - + 0 - 0 + \cdots$

+ and − alternate with arbitrary number of 0's in between them. → (hidden AF order)

$\tilde{\mathcal{H}}$: restricted Hilbert space generated by these basis states

Theorem (Tasaki '86 unpublished)

The Heisenberg AF on $\tilde{\mathcal{H}}$ has a unique g.s. with a gap and exponentially decaying correlation function

⟨AKLT model and the VBS picture⟩

§ AKLT model for $S=1$

$S=1$ (AF) chain with

$$\hat{H}_{AKLT} = \sum_{x=1}^{L} \left\{ \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \frac{1}{3} (\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1})^2 \right\}$$

still AF, and SU(2) invariant

<u>Theorem</u> (Affleck-Kennedy-Lieb-Tasaki 1987)

- The g.s. is unique (for finite and infinite L)
- $\exists$ a nonvanishing energy gap (uniform in L)
- $\langle \Phi_{GS}, \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_{GS} \rangle = (-1)^{|x-y|} 4 \cdot 3^{-|x-y|}$ $(|x-y| \geq 2)$

$\exists$ gap in the infinite system
Matsui

strong support to the Haldane conjecture
→ <u>BUT NOT A PROOF!!</u>

a stability theorem (very difficult but important)

<u>Theorem</u> (Yarotsky 2006)

$\hat{V}$: any short ranged translation invariant interaction

$\hat{H}_\varepsilon = \hat{H}_{AKLT} + \varepsilon \hat{V}$ For suff. small ε,

the g.s. is unique, $\exists$ a gap, exp. decay.

§VBS (valence-bond-solid) state

exact g.s. of the AKLT model

$$\hat{S}_x \cdot \hat{S}_{x+1} + \frac{1}{3}(\hat{S}_x \cdot \hat{S}_{x+1})^2 = 2\hat{P}_2(\hat{S}_x + \hat{S}_{x+1}) - \frac{2}{3} \quad \text{—} \circledast$$

the e.v. of $(\hat{S}_x + \hat{S}_{x+1})^2 \to S'(S'+1)$ with $S' = 0, 1, 2$.

$\hat{P}_2(\hat{S}_x + \hat{S}_{x+1})$: the proj. onto the space with $S' = 2$

$\hat{H}_{AKLT}$ is essentially the same as

$$\hat{H}'_{AKLT} = \sum_{x=1}^{L-1} \hat{P}_2(\hat{S}_x + \hat{S}_{x+1})$$

We shall construct Φ_{VBS} s.t. $\hat{P}_2(\hat{S}_x + \hat{S}_{+1})\, \Phi_{VBS} = 0$ for $\forall x$.

Then it is a g.s. of $\hat{H}'_{AKLT}$ (and $\hat{H}_{AKLT}$)

Rem.
This whole symbol denotes the projector NOT $\hat{P}_2$ time $(\hat{S}_x + \hat{S}_{x+1})$

VBS-1 Show $\circledast$

construction of the VBS state

- two $S=\frac{1}{2}$'s. L R $\quad \psi_L^{\sigma} \otimes \psi_R^{\sigma'} \quad \sigma, \sigma' = \uparrow, \downarrow$

$\hat{\mathcal{S}}$ → symmetrization

$$\hat{\mathcal{S}}(\psi_L^{\sigma} \otimes \psi_R^{\sigma'}) = \underbrace{\frac{1}{2}\{\psi_L^{\sigma} \otimes \psi_R^{\sigma'} + \psi_L^{\sigma'} \otimes \psi_R^{\sigma}\}}_{\text{total spin } 1.}$$

→ projection op. onto the supspace with $S_{tot}=1$.

- duplicated chain. with sites $(x, L), (x, R)$ $\quad x=1, \cdots, L$

$x-1$ x $x+1$ ····
L R   L R   L R

put $S=\frac{1}{2}$'s on each site

$$\Phi_{\text{pre-VBS}} := \bigotimes_{x=1}^{L} \underbrace{\frac{1}{\sqrt{2}}\{\psi_{x,R}^{\uparrow} \otimes \psi_{x+1,L}^{\downarrow} - \psi_{x,R}^{\downarrow} \otimes \psi_{x+1,L}^{\uparrow}\}}$$

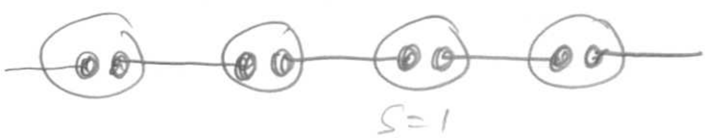
singlet pair = valence-bond

a state for $2L$ spin $\frac{1}{2}$'s.

$$\Phi_{VBS} := \left(\bigotimes_x \hat{\mathcal{S}}_x\right) \Phi_{\text{pre-VBS}}$$

valence-bond solid state

a state for the $S=1$ chain.



BUT note that

$$\hat{P}_2(\hat{\mathbf{S}}_x+\hat{\mathbf{S}}_{x+1})\,\Phi_{VBS} = \hat{P}_2(\hat{\mathbf{S}}_x+\hat{\mathbf{S}}_{x+1})\left(\bigotimes_x \hat{\mathcal{S}}_x\right)\Phi_{\text{pre-VBS}}$$

$$= \left(\bigotimes_x \mathcal{S}_x\right) \underbrace{\hat{P}_2(\hat{\mathbf{S}}_{x,L}+\hat{\mathbf{S}}_{x,R}+\hat{\mathbf{S}}_{x+1,L}+\hat{\mathbf{S}}_{x+1,R})\,\Phi_{\text{pre-VBS}}}_{=\,0}$$

singlet!

Φ_{VBS} is an exact g.s. of $\hat{H}_{AKLT}$

The theorem is proved based on the exact g.s. and the special properties of the model.

gap: a simpler proof Knabe 88

$\downarrow$

(general theory Fannes, Nachtergaele, Werner 92)

See also Matsui

§ VBS in the standard basis — hidden AF order + ~~MPS~~ intro to MPS

$$\bullet\!\!-\!\!\bullet = \frac{1}{\sqrt{2}}\left\{ (\uparrow \overset{①}{—} \downarrow) - (\downarrow \overset{②}{—} \uparrow) \right\}$$

$$\overset{①}{—}\downarrow\uparrow\overset{①}{—}\downarrow\downarrow\overset{②}{—}\uparrow\downarrow\overset{②}{—}\uparrow\uparrow\overset{①}{—}\downarrow\uparrow\overset{①}{—}\downarrow\uparrow\overset{①}{—}\downarrow\downarrow\overset{②}{—}\uparrow\uparrow\overset{①}{—}$$

$$0 \quad - \quad 0 \quad + \quad 0 \quad 0 \quad - \quad +$$

+ and − alternate with arbitrary numbers of 0's in between them!

- "quantum spin liquid" with hidden AF order

- the same expansion whatever "quantization axis" is taken
  - standard AF order → appears in a specific direction
  - hidden AF order → appears in any directions !!

note

$$\begin{cases} \mathcal{S}(\uparrow\uparrow) = \psi^{+} \\ \mathcal{S}(\downarrow\downarrow) = \psi^{-} \\ \mathcal{S}(\uparrow\downarrow) = \frac{1}{2}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{\sqrt{2}}\psi^{0} \end{cases}$$

Matrix product representation

Fannes, Nachtergaele, Werner 89
Klümpen, Schadschneider, Zittarz 91

$$\Phi_{VBS} = \sum_{\sigma}{}' C_{\sigma} \Psi^{\sigma}$$

standard basis

coefficients.

$\sigma = (\sigma_x)_{x=1,\dots,L}$

$\sigma_x = 0, \pm$

$$C_{\sigma_1,\dots,\sigma_L} = \sum_{\alpha_1,\dots,\alpha_L=1,2\sim D}{}' A_{\sigma_1 \alpha_1 \alpha_2} A_{\sigma_2 \alpha_2 \alpha_3} \cdots A_{\sigma_L \alpha_L \alpha_1}$$

$$= \mathrm{Tr}[A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_L}]$$

$A_\sigma = (A_{\sigma\alpha\alpha'})_{\alpha,\alpha'=1,2}$

2×2 matrix

$\downarrow \overset{②}{—} \uparrow\uparrow \overset{①}{—} \downarrow$ $\quad +$ $\qquad A_{+21} = -\frac{1}{\sqrt{2}}$

$\uparrow \overset{①}{—} \downarrow\downarrow \overset{②}{—} \uparrow$ $\quad -$ $\qquad A_{-12} = \frac{1}{\sqrt{2}}$

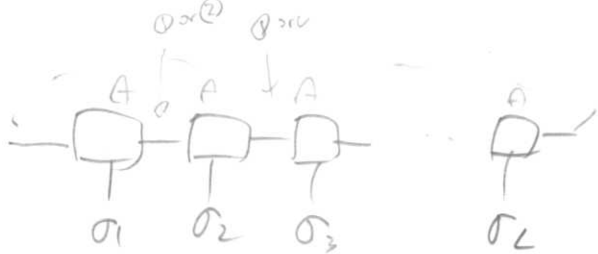
$\uparrow \overset{①}{—} \downarrow\uparrow \overset{①}{—} \downarrow$ $\quad 0$ $\qquad A_{011} = \frac{1}{2}$

$\downarrow \overset{②}{—} \uparrow\downarrow \overset{②}{—} \uparrow$ $\quad 0$ $\qquad A_{022} = -\frac{1}{2}$

$A_{\sigma,\alpha\alpha'} = 0$ otherwise.

$$A_+ = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad A_- = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \quad A_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$C_{\sigma_1,\dots,\sigma_L} =$



generalization
- Finitely correlated states   Fannes ...
- Matrix product states (MPS)

- represent a large class of states in 1 dim with small entanglement. → " ~~[illegible]~~

- dense in the ~~t~~transl. invariant states

BUT STILL SPECIAL!!

application
computation of normalization.

$$\langle \Phi_{VBS}, \Phi_{VBS} \rangle = \sum_{\sigma}{}' (C_\sigma)^2 = \sum_{\sigma} \sum_{\substack{\alpha_1 \cdots \alpha_L \\ \alpha_1' \cdots \alpha_L'}}{}' A_{\sigma_1 \alpha_1 \alpha_2} \cdots A_{\sigma_L \alpha_L \alpha_1} A_{\sigma_1 \alpha_1' \alpha_2'} \cdots A_{\sigma_L \alpha_L' \alpha_1'}$$

$$= \sum_{\substack{\alpha_1 \cdots \alpha_L \\ \alpha_1' \cdots \alpha_L'}}{}' \tilde{A}_{\alpha_1 \alpha_1' \alpha_2 \alpha_2'} \tilde{A}_{\alpha_2 \alpha_2' \alpha_3 \alpha_3'} \cdots \tilde{A}_{\alpha_L \alpha_L' \alpha_1 \alpha_1'}$$

$\tilde{A}$: 4×4 matrix

$$= \mathrm{Tr}[\tilde{A}^L]$$

also correlation

MPS-1 Compute $\langle \Phi_{VBS}, \Phi_{VBS} \rangle$ explicity ($S=\frac{1}{2}$ chain)

MPS-2 What is the MP rep. of $(\Phi_\uparrow + \Phi_\downarrow)$ ↑↑↑↑... ↓

MPS-3 What is the MP rep. of $\hat{S}^- \Phi_\uparrow$ ↑↑↓↑↑↑

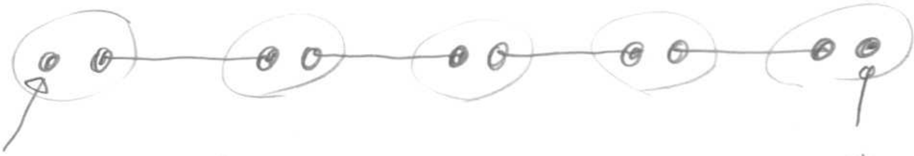
try $C_\sigma = \sum_{\alpha_1 \cdots \alpha_L = 1, \cdots, D} A_{\sigma_1 \alpha_1 \alpha_2} \cdots A_{\sigma_L \alpha_L \alpha_1}$ what is D??

and $C_\sigma = \sum_{\alpha_1 \cdots, \alpha_{L+1} = 1, \cdots, D} L_{\alpha_1} A_{\sigma_1 \alpha_1 \alpha_2} \cdots A_{\sigma_L \alpha_L \alpha_{L+1}} R_{\alpha_{L+1}}$

§ VBS states open chains — edge states

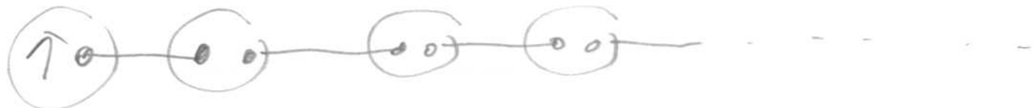
AKLT model on periodic chain, infinite chain
→ the g.s. is unique.

on an open chain



There are four ground states

~~Semi-infinite~~ long chain with extra ↑

$\Phi'_{VBS} =$ 

the edge spin is not completely localized

$$\langle \Phi'_{VBS}, \hat{S}^{(1)}_x \Phi'_{VBS} \rangle = \langle \Phi'_{VBS}, \hat{S}^{(2)}_x \Phi'_{VBS} \rangle = 0$$

$$\langle \Phi'_{VBS}, \hat{S}^{(3)}_x \Phi'_{VBS} \rangle = -2(-3)^{-x}$$

$$\sum_{x=1}^{\infty} \langle \quad \rangle = \frac{1}{2}$$

$$\Lambda_L = \left\{ -\frac{L}{2}+1, -\frac{L}{2}+2, \cdots, \frac{L}{2} \right\}$$

the four g.s. converges to a single inf. vol. gr. st. as $L \uparrow \infty$

recall that

	finite L	$L \uparrow \infty$
Heisenberg AF $d \geq 2$	unique g.s.	infinitely many "g.s."
AKLT open chain	four g.s.	unique "g.s."

§ VBS picture

Can we form VB(S) states for other S ?

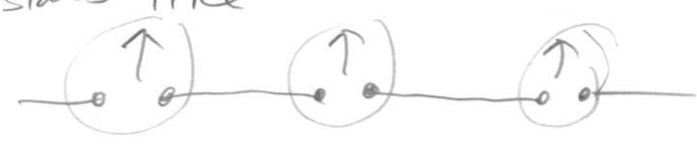
$S=2$  four $S=\frac{1}{2}$'s

$S=\frac{3}{2}$  three $S=\frac{1}{2}$'s

translation inv. is broken.

$\boxed{\text{We can construct translation invariant VBS only for integer } S.}$

BUT under magnetic field one may have a g.s. states like



for $S=\frac{3}{2}$ ~~sp~~ chain

VBS like state

here $S^{(3)}_{TOT}=\frac{L}{2}$

YOA filling factor is $\nu=\frac{1}{2}+\frac{3}{2}=\underline{2}$ integer!

Skip

⟨Haldane phase⟩

§ Haldane conj. for the $S=1$ Heisenberg AF chain

$$\hat{H} = \sum_{x=1}^{L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1}$$

numerical results → gap is also observed experimentally!

- $\exists$ a gap $\simeq 0.41$ above the unique g.s.
- correlation in g.s. decays exponentially

BUT STILL NO PROOF → AKLT is at the "center" of the "Haldane phase", and the Heisenberg AF happens to belong to that phase ? ? ?

§. The model with anisotropy

$S=1$ chain (pbc)

$$\hat{H}_{\text{aniso}} = \sum_{x=1}^{L} \{ \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + \underbrace{D(\hat{S}_x^{(3)})^2}_{\text{anisotropy}} \} \qquad D \geq 0$$

note that

$$\hat{H}_0 = \sum_{x=1}^{L} D(\hat{S}_x^{(3)})^2 \quad \text{is trivial}$$

G.S. $\Phi_0 = \bigotimes_{x=1}^{L} \varphi_x^0$ 　　 0 0 0 0 0 0 0

$E_0 = 0$

1st excited 　 0 0 0 + 0 0 0 　 or 　 0 0 0 − 0 0 0

$E_{1st} = E_0 + \underbrace{D}_{\text{energy gap}}$

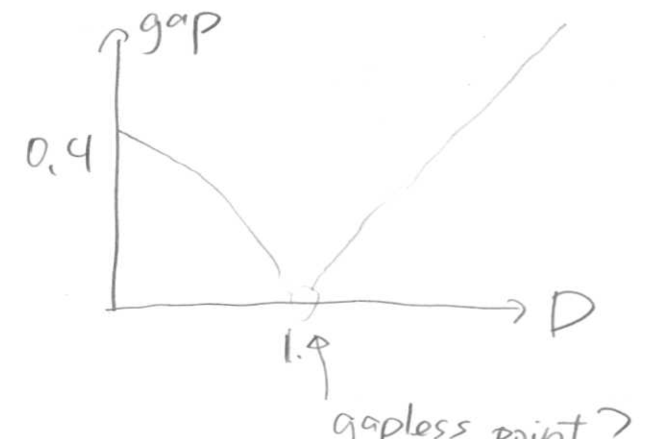
if $D \gg 1$

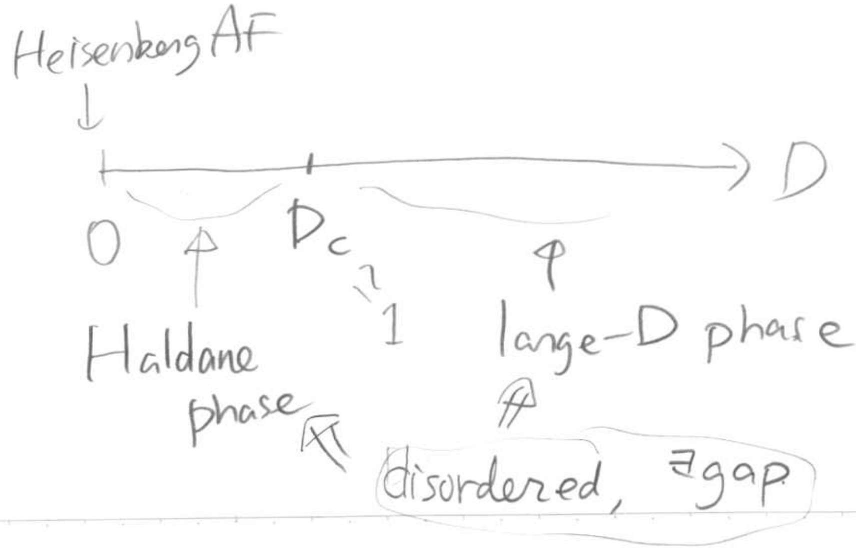
- The g.s. is unique and is close to Φ_0
- $\exists$ a gap $\simeq D$
- The g.s. correlation decays exponentially

all rigorous and trivial
(cluster expansion)

Is the Haldane gap smoothly connected to this trivial gap?

numerical results





§ Peculiar features of the Haldane phase

Hidden AF order

The g.s. of $\hat{H}_{\mathrm{aniso}}$ $\quad \Phi_{GS} = \sum_{\sigma \, (\sum_x \sigma_x = 0)} C_\sigma \tilde{\Psi}^{\sigma}$ $\quad$ for $\forall D$

$C_\sigma > 0$ for $\forall \sigma$ (Marshall-Lieb-Mattis)

(different from the VBS state)

BUT in the Haldane phase. most states (with considerable weight) look like

$+ \; 0 \; - \; + \; - \; 0 \; 0 \; - \; 0 \; + \; 0 \; + \; 0 \; - \; + \; -$

defect

the long-range hidden AF order still presents

$+ \; - \; + \; - \; - \; + \; + \; - \; + \; - \; +$

even $\qquad\qquad$ even

denNijs-Rommelse string order parameter 1989

$$O^{(\alpha)}_{\mathrm{string}} := -\lim_{|x-y|\uparrow\infty} \lim_{L\uparrow\infty} \Big\langle \Phi_{GS}, \hat{S}^{(\alpha)}_x \underbrace{\exp\Big[i\pi \sum_{z=x+1}^{y-1} \hat{S}^{(\alpha)}_z\Big]}_{= (-1)^{\sum \hat{S}^{(\alpha)}_z}} \hat{S}^{(\alpha)}_y \Phi_{GS} \Big\rangle$$

$\alpha = 1, 2, 3$

for the VBS state $O^{(\alpha)}_{\text{string}} = \frac{4}{9}$ $\alpha = 1, 2, 3.$

heuristic arguments
+ numerical res. for $\hat{H}_{\text{aniso}}$

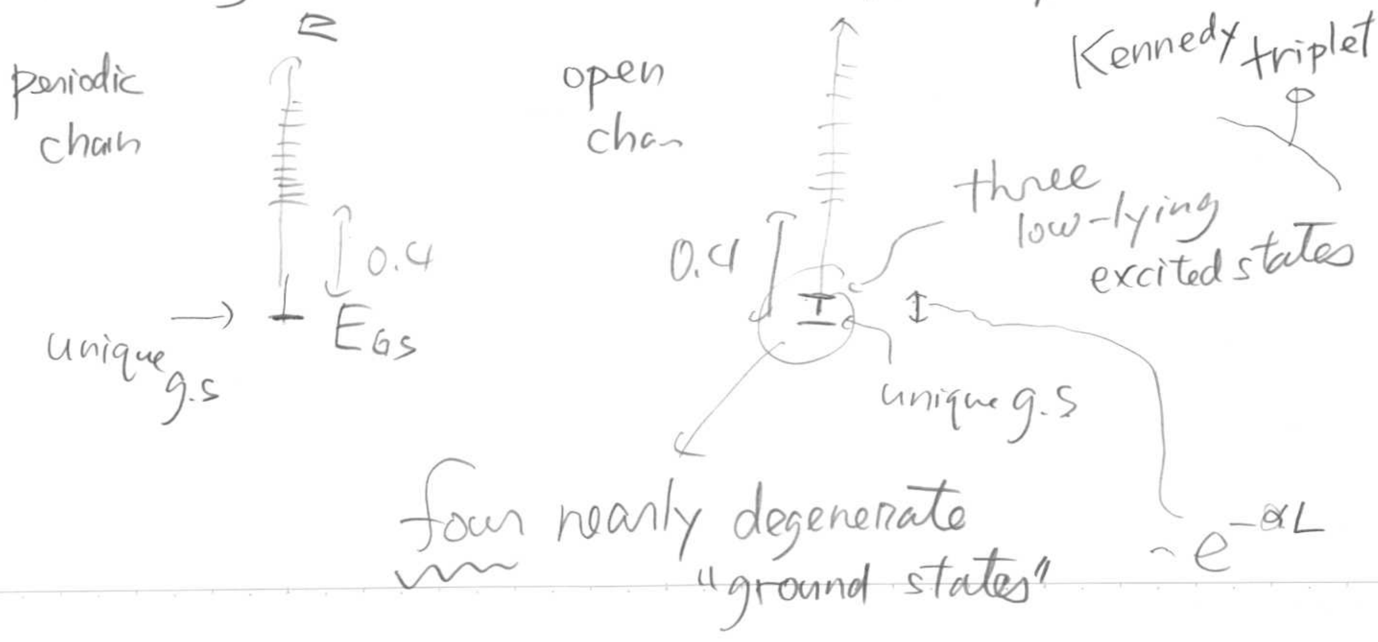
$$\begin{cases} \text{Haldane phase} & O^{(1)}_{\text{strg}} = O^{(2)}_{\text{strg}} > 0, \quad O^{(3)}_{\text{string}} > 0 \\ \text{large-}D \text{ phase} & O^{(1)}_{\text{strg}} = O^{(2)}_{\text{strg}} = O^{(3)}_{\text{strg}} = 0 \end{cases}$$

$\boxed{\text{The hidden AF order (measured by the string order par.) characterizes the Haldane phase.}}$

Near four-fold degeneracy and the edge state

- AKLT model on $\begin{cases} \text{a periodic chain} \to \text{unique g.s. + a gap} \\ \text{an open chain} \to \text{four g.s. + a gap} \end{cases}$

- Heisenberg AF (numerical) Kennedy 1990



hidden AF order $\Longrightarrow$ near four-fold degeneracy
for open chain

1) Harsch-von den Liden Theorem

$$\hat{O}^{(\alpha)}_{string} := \sum_{x=1}^{L} \hat{S}_x^{(\alpha)} \exp\left[i\pi \sum_{y=1}^{x-1} \hat{S}_y^{(\alpha)}\right]$$

if $O^{(\alpha)}_{string} \neq 0$

Then $\langle \Phi_{GS}, (\hat{O}^{(\alpha)}_{string})^2 \Phi_{GS} \rangle \geq \alpha . L^2$ $\quad \alpha > 0$

Thus $\dfrac{\hat{O}^{(\alpha)}_{string} \Phi_{GS}}{\| \hat{O}^{(\alpha)}_{string} \Phi_{GS} \|}$ is a low-lying state

$\alpha = 1, 2, 3$

they are orthogonal

2) 0, +, − configuration

config. with complete hidden AF order

$$\left.\begin{array}{l} |0 + 0 - + \cdots\cdots - + 0 0 -| \\ |0 0 - + \cdots\cdots + | \\ |- \cdots\cdots - | \\ |+ \cdots\cdots + | \end{array}\right\} \text{four kinds}$$

edges states

Thus. "Haldane phase" is a distinct phase

0 Haldane $D_c \simeq 1$ large-D $\to D$

Haldane	large-D
hidden AF order	no order
near-four-fold degeneracy in open chain (edge states)	unique GS with a gap in open chain.

quite exotic!

observed experimentally!

§ Non-local unitary transformation and hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking

(Kennedy-Tasaki 92)

open chain

$$\hat{H} = \sum_{x=1}^{L-1} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_{x+1} + D \sum_{x=1}^{L} (\hat{S}_x^{(3)})^2$$

basis state $\Psi^{\sigma} = \bigotimes_{x=1}^{L} \psi_x^{\sigma_x}$ with $\sigma = (\sigma_x)_{x=1,\cdots,L}$
$\sigma_x = 0, \pm 1$.

For σ, define $\sigma' = (\sigma'_x)_{x=1,\cdots,L}$ by

$$\sigma'_x = \underbrace{(-1)^{\sum_{y=1}^{x-1} \sigma_y}} \sigma_x$$

σ 0 0 + 0 − + 0 + − 0 − + hidden AF order

σ' 0 0 + 0 + + 0 − − 0 + + ferro order

local defect

Define unitary op. $\hat{U}$ by

$$\hat{U} \Psi^{\sigma} = (-1)^{N(\sigma)} \Psi^{\sigma'}$$

$N(\sigma)$: the number of odd x with $\sigma_x = 0$.

$$\left(\text{Oshikawa's form} \quad \hat{U} = \prod_{x<y} \exp[i\pi \hat{S}_x^{(3)} \hat{S}_y^{(1)}] \right)$$

$$\left(\begin{aligned} &\hat{H}\,\Phi_{GS} = E\,\Phi_{GS} \\ &\hat{H}'\,\Phi'_{GS} = E\,\Phi'_{GS} \qquad \Phi'_{GS} = \hat{U}\,\Phi_{GS} \end{aligned} \right)$$

Then

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^{\dagger}$$

$$= \sum_{x=1}^{L-1} \left\{ -\underbrace{\hat{S}^{(1)}_x \hat{S}^{(1)}_{x+1}} + \hat{S}^{(2)}_x e^{i\pi(\hat{S}^{(3)}_x + \hat{S}^{(3)}_{x+1})} \hat{S}^{(2)}_{x+1} - \underbrace{\hat{S}^{(3)}_x \hat{S}^{(3)}_{x+1}} \right\}$$

$$+ D \sum_{x=1}^{L} \left(\hat{S}^{(3)}_x\right)^2$$

is

- mainly ferromagnetic (especially in the 1st and the 3rd directions)

- has a discrete symmetry

  invariant under the π-rotation around the 1, 2, or 3 axis.

  not independent.

  $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

The long-range order parameters of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking

$$O^{(\alpha)}_{\text{ferro}} = \lim_{|x-y|\uparrow\infty} \langle \Phi'_{GS}, \hat{S}^{(\alpha)}_x \hat{S}^{(\alpha)}_y \Phi'_{GS} \rangle \qquad (\alpha = 1, 3)$$

$$\Phi'_{GS} = \hat{U}\,\Phi_{GS}$$

then it holds that

$$\boxed{O^{(\alpha)}_{\text{ferro}} = O^{(\alpha)}_{\text{string}} \qquad (\alpha = 1, 3)}$$

Φ'_{GS} $\qquad\qquad$ Φ_{GS}

The picture of hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking

$\hat{H}'$: ferromagnetic Hamiltonian with discrete $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

- large-D phase $D > D_c$
  no symmetry breaking
  unique g.s. + a gap

- Haldane phase $0 \leq D < D_c$
  $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is fully broken
  - SSB of a discrete symmetry $\rightarrow$ gap
  - ferromagnetic order $\rightarrow$ hidden AF order
  - four g.s. in the infinite chain
    ↳ four low-lying energy excitations in a finite chain

($\hat{H}$ and $\hat{H}'$ have exactly the same spectra)

all the exotic properties of the Haldane phase can be understood as a consequence of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking.

$\rightarrow$ starting point of other rigorous and non-rigorous theories

⟨Some related issues⟩

§ Stability of the Haldane phase

Does the $\mathbb{Z}_2 \times \mathbb{Z}_2$ picture explain everything?

- edge states of the $S=2$ VBS

$3 \times 3 = 9$ fold degeneracy.

($\mathbb{Z}_2 \times \mathbb{Z}_2$ suggests four) ?

- string order for the general VBS (Oshikawa 92)

$$O^{(\alpha)}_{\mathrm{string}} \begin{cases} > 0 & \text{for } S = 1, 3, 5, \cdots \\ = 0 & \text{for } S = 2, 4, 6, \cdots \end{cases}$$?

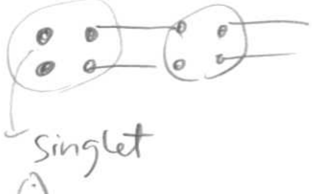
Is it possible to connect the Haldane and the large-D phases smoothly ?

Is there $\hat{H}_\lambda$ such that

- $\hat{H}_\lambda$: Ham on the open chain, depends smoothly on $\lambda \in [0,1]$ has a suitable symmetry (e.g. inv. under $\hat{\mathbf{S}}_x \to -\hat{\mathbf{S}}_x$ for all x)
- $\hat{H}_0 = \sum_{x=1}^{L} D(\hat{S}_x^{(3)})^2$, $\hat{H}_1 = \hat{H}_{AKLT}$
- $\hat{H}_\lambda$ has a unique g.s. + a gap for $\forall \lambda \in [0,1]$?

Yes for $S = 2, 4, 6, \cdots$

edge



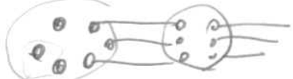
singlet

No for $S = 1, 3, 5, \cdots$

(Haldane phase is a "symmetry protected topological phase")

but any phase is protected by a symmetry

edge

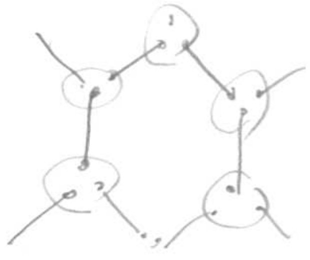


at least $S = \frac{1}{2}$ remains ($\to$ four-fold near degeneracy)

Gu, Wen
Pollmann, Turner, Berg, Oshikawa

§ VBS in two-dimensions

$S=\frac{3}{2}$ model on the hexagonal lattice



- g.s. is unique
- g.s. correlation decay exponentially

Tensor network

The same game.

PEPS

$A_{\alpha\alpha'\alpha''}$

no proof of gap

hidden order ??

SSB ???

no ideas

§ Hamiltonian vs. states

(ground) states are more important than the Ham. → MPS

VBS, Laughlin, BCS ....

tensor network, ...

start from states

New TREND?!

BUT philosophically

microscopic physics → low energy effective theories → g.s. or eq. states with intersting physics

we miss these exciting steps!

practically We miss many important problems (Haldane gap in $S=1$ Heisenberg AF)