Part 1

Long-range order (LRO) and Spontaneous symmetry breaking (SSB)

LRO and SSB appear universally in a wide range of systems with large (infinite) degrees of freedom

finite temp. (equilibrium) { · classical · quantum → ~~similar~~ nontrivial interesting

ground state · classical → trivial

· quantum $[\hat{H}, \hat{O}] = 0$ → trivial

$[\hat{H}, \hat{O}] \neq 0$ → nontrivial interesting ← MAIN TOPIC

→ antiferro., BEC

field theory

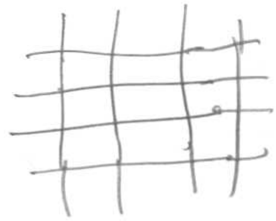
⋮

$\hat{H}$: Hamiltonian

$\hat{O}$ = order parameter

⟨Some results about the Ising model⟩

§ Definitions



$L \times \cdots \times L$ d-dim. hypercubic lattice.

$(\Lambda_L, \mathcal{B}_L)$ set of bonds, set of sites

L even

$$\Lambda_L := \left\{ (x_1, \dots, x_d) \,\middle|\, x_i \in \mathbb{Z}, -\frac{L}{2} < x_i \leq \frac{L}{2} \right\} \subset \mathbb{Z}^d$$

$$\mathcal{B}_L := \left\{ (x, y) \,\middle|\, x, y \in \Lambda_L, |x - y| = 1 \right\}$$

use periodic b.c.

$(x, y) = (y, x)$

<u>Spin variables</u> $\sigma_x = \pm 1, \; x \in \Lambda_L$

$$\boldsymbol{\sigma} = (\sigma_x)_{x \in \Lambda} \in \{-1, 1\}^{\Lambda_L}$$

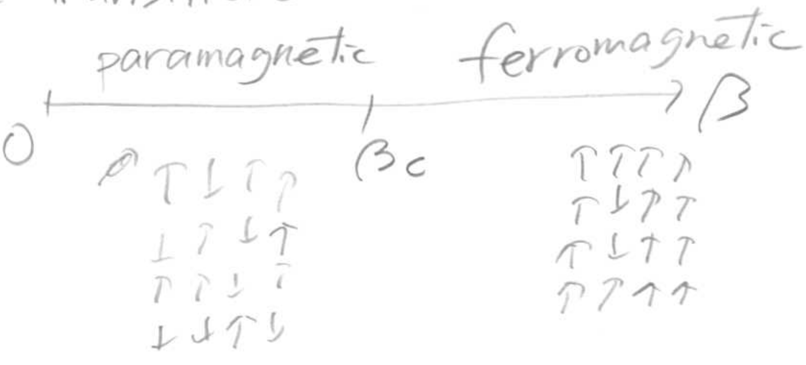
<u>Hamiltonian</u> $H(\boldsymbol{\sigma}) := -\sum_{(x,y) \in \mathcal{B}_L} \sigma_x \sigma_y$ omit

<u>thermal equilibrium at $\beta > 0$</u>

$$\begin{cases} \langle \cdots \rangle_{\beta, L} := \{Z_L(\beta)\}^{-1} \sum_{\boldsymbol{\sigma}} (\cdots) e^{-\beta H} \\ Z_L(\beta) := \sum_{\boldsymbol{\sigma}} e^{-\beta H} \end{cases}$$

§ Phase transition

$d \geq 2$



Behavior of correlation function

$$\begin{cases} \beta < \beta_c & \langle \sigma_x \sigma_y \rangle_{\beta,L} \leq e^{-\frac{|x-y|}{\xi(\beta)}} \quad \xi(\beta) > 0 \\ & \text{exponential decay} \\ \beta > \beta_c & \langle \sigma_x \sigma_y \rangle_{\beta,L} \geq q(\beta) > 0 \quad \text{for } \forall x, y \end{cases}$$

does not decay!

$\downarrow$

long-range order (LRO)

$$q(\beta) := \lim_{|x-y| \uparrow \infty} \lim_{L \uparrow \infty} \langle \sigma_x \sigma_y \rangle_{\beta,L}$$

§ LRO and SSB

- relevant symmetry global spin flip $\sigma \to -\sigma$
- order parameter $\Theta := \sum_{x\in\Lambda_L} \sigma_x$ (total magnetization)

$$\left\langle \left(\frac{\Theta}{L^d}\right)^2 \right\rangle_{\beta,L} = L^{-2d} \sum_{x,y\in\Lambda_L} \langle \sigma_x \sigma_y \rangle_{\beta,L}$$

$$\underset{\text{transl. inv.}}{=} L^{-d} \sum_{x\in\Lambda_L} \langle \sigma_0 \sigma_x \rangle_{\beta,L} \begin{cases} = O(L^{-d}) & \beta<\beta_c \quad \text{NO ORDER.} \\ \gtrsim q(\beta) > 0 & \beta > \beta_c \quad \text{LRO} \end{cases}$$

BUT, from the symmetry

$$\langle \Theta \rangle_{\beta,L} = 0 \quad \text{for } \forall \beta, L$$

NO SSB ?? ~~(BUT LRO)~~ $\boxed{\text{LRO without SSB}}$

the state is "unphysical" because

$$(\text{fluctuation of } \frac{\Theta}{L^d}) = \sqrt{\left\langle \left(\frac{\Theta}{L^d}\right)^2 \right\rangle - \left\langle \frac{\Theta}{L^d} \right\rangle^2} \gtrsim \sqrt{q(\beta)} > 0$$

if $\beta > \beta_c$

macroscopic quantity ~~has huge~~ exhibits fluctuation!

Hamiltonian with symmetry breaking field

$$H_h = \underbrace{-\sum_{x,y\in\mathcal{B}_L} \sigma_x\sigma_y}_{H} - h \underbrace{\sum_{x\in\Lambda_L}\sigma_x}_{=\,\mathcal{O}}$$

$$\begin{cases} \langle \cdots \rangle_{\beta,h;L} := Z_L(\beta,h)^{-1} \sum_{\sigma} (\cdots)\, e^{-\beta H_h} \\ Z_L(\beta,h) := \sum_{\sigma} e^{-\beta H_h} \end{cases}$$

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty} \left\langle \frac{\mathcal{O}}{L^d} \right\rangle_{\beta,h;L} = \begin{cases} 0 & \beta<\beta_c \\ \sqrt{q(\beta)} & \beta>\beta_c \end{cases}$$

Of course

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty} \left\langle \left(\frac{\mathcal{O}}{L^d}\right)^2 \right\rangle_{\beta,h;L} = \begin{cases} 0 & \beta<\beta_c \\ q(\beta) & \beta>\beta_c \end{cases}$$

$\boxed{\text{LRO} + \text{SSB}}$

healthy!

(fluct. of $\frac{\mathcal{O}}{L^d} \to 0$ as $L\uparrow\infty$)

⟨Basics about quantum spin systems⟩

§ Some elementary linear algebra

- positive semidefinite operator (matrix)

$\mathcal{H}$: a finite dim. Hilbert space

$\hat{A}$: hermitian operator on $\mathcal{H}$

$$\hat{A} \geq 0 \Leftrightarrow \langle \Phi, \hat{A}\Phi \rangle \geq 0 \text{ for } {}^\forall \Phi \in \mathcal{H}$$

$$\Leftrightarrow \text{min. e.v. of } \hat{A} \geq 0$$

$\hat{A}, \hat{B}$ hermitian $\quad \hat{A} - \hat{B} \geq 0 \Leftrightarrow \hat{A} \geq \hat{B}$

Th. $\hat{A} \geq 0,\ \hat{B} \geq 0 \Rightarrow \hat{A} + \hat{B} \geq 0$ (we don't assume $[\hat{A}, \hat{B}] = 0$)

$$\because \langle \Phi, (\hat{A}+\hat{B})\Phi \rangle \geq \langle \Phi, \hat{A}\Phi \rangle + \langle \Phi, \hat{B}\Phi \rangle \geq 0$$

Corollary. Let $\hat{H} = \sum_j{}' \hat{H}_j$, and assume $\hat{H}_j \geq \epsilon_j$. If Φ satisfies $\hat{H}_j \Phi = \epsilon_j \Phi$ for ${}^\forall j$ then Φ is a ground state of $\hat{H}$.

$\because \hat{H} \geq (\sum_j \epsilon_j)$ and $\hat{H}\Phi = (\sum_j \epsilon_j)\Phi$

simultaneously mimimizable

("frustration free")

This will be used repeatedly. (Too much)

- operator norm $\quad \hat{A}$ any operator

$$\|\hat{A}\| := \max_{\Phi \text{ s.t. } \|\Phi\| = 1} \frac{\|\hat{A}\Phi\|}{\|\Phi\|}$$

one has $\|\hat{A}\hat{B}\| \leq \|\hat{A}\| \, \|\hat{B}\|$

• Perron-Frobenius theorem

$n\times n$ matrix $A=(a_{ij})_{i,j=1,\cdots,n}$

i) $a_{ij}\in\mathbb{R}$

ii) $a_{ij}\le 0$ if $i\ne j$

iii) $\forall i\ne j$ are connected via nonvanishing elements of A

i.e. $\exists i_1,\dots,i_k$

s.t. $i_1=i,\ i_k=j,\ a_{i_\ell i_{\ell+1}}\ne 0\ (\ell=1,\dots,k-1)$

Theorem Assume i), ii), iii), then $\exists$ a real nondegenerate e.v. λ_{PF} of A, and the corresponding eigenvector $v=(v_1,\cdots,v_n)$ can be taken to satisfy $v_i>0$. We have $\lambda_{PF}<\mathrm{Re}\,\lambda$ for any eigenvalue $\lambda\ne\lambda_{PF}$

(proof → see my book
elementary, but not easy)

If A is real symmetric, λ_{PF} is the lowest eigenvalue
(ground state energy)

↓

proof of the theorem is easy.

($v_i>0$
the g.s. wave function is "nodeless")

§ Quantum spin systems – general definition and properties

- general lattice Λ
- spin $S = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ ← fix

spin at site $x \in \Lambda$

$$\mathcal{H}_x = \mathbb{C}^{2S+1} \quad \text{the Hibert space at } x$$

$$\hat{\mathbf{S}}_x = (\hat{S}_x^{(1)}, \hat{S}_x^{(2)}, \hat{S}_x^{(3)}) \quad \text{spin operator at } x$$

$$[\hat{S}_x^{(\alpha)}, \hat{S}_x^{(\beta)}] = i \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \hat{S}_x^{(\gamma)}$$

$$(\hat{\mathbf{S}}_x)^2 = \sum_{\alpha=1}^{3} (\hat{S}_x^{(\alpha)})^2 = S(S+1)$$

$$\hat{S}_x^{\pm} := \hat{S}_x^{(1)} \pm i \hat{S}_x^{(2)}$$

basis states $\psi_x^{(\sigma)} \in \mathcal{H}_x$ $\quad \sigma = -S, -S+1, \cdots, S$

$$\begin{cases} \hat{S}_x^{(3)} \psi_x^{(\sigma)} = \sigma \psi_x^{(\sigma)} \\ \hat{S}_x^{\pm} \psi_x^{(\sigma)} = \sqrt{S(S+1) - \sigma(\sigma \pm 1)}\, \psi_x^{(\sigma \pm 1)} \end{cases}$$

$\sqrt{2S}$

$$S^- \psi^{(S)} = \sqrt{S(S+1) - S(S-1)}\, \psi^{(S-1)}$$

quantum spin system on Λ

$$\mathcal{H} := \bigotimes_{x\in\Lambda} \mathcal{H}_x \quad \text{\# whole Hibert space}$$

basis states $\Psi^{\sigma} := \bigotimes_{x\in\Lambda} \psi_x^{\sigma_x}$

spin config. $\sigma = (\sigma_x)_{x\in\Lambda}$, $\quad \sigma_x = -S, -S+1, \cdots, S$

$\hat{S}_x^{(\alpha)}$ acts on $\psi_x^{\sigma_x}$

total spin

$$\hat{\mathbf{S}}_{tot} := \sum_{x\in\Lambda} \hat{\mathbf{S}}_x = (\hat{S}_{tot}^{(1)}, \hat{S}_{tot}^{(2)}, \hat{S}_{tot}^{(3)})$$

$$\hat{S}_{tot}^{\pm} := \hat{S}_{tot}^{(1)} \pm i \hat{S}_{tot}^{(2)}$$

The eigenvalues of $(\hat{\mathbf{S}}_{tot})^2$ is denoted as

$$S_{tot}(S_{tot}+1)$$

with $S_{tot} \in \{|\Lambda|S, |\Lambda|S-1, \cdots, \frac{1}{2} \text{ or } 0\}$

THIS IS WRONG

QS-0: Correct this mistake.

properties of $\mathbb{S}_x \cdot \mathbb{S}_y$ ← building block of the Heisenberg model

(the most natural model for interacting spins) → Part 3

$\boxed{\forall S}$

$$[\hat{\mathbb{S}}_x \cdot \hat{\mathbb{S}}_y, \hat{S}^{(\alpha)}_{tot}] = 0 \qquad \alpha = 1, 2, 3$$

$$\hat{\mathbb{S}}_x \cdot \hat{\mathbb{S}}_y = \frac{1}{2}\{(\hat{\mathbb{S}}_x + \hat{\mathbb{S}}_y)^2 - \hat{\mathbb{S}}_x^2 - \hat{\mathbb{S}}_y^2\}$$

$$= \frac{1}{2}(\hat{\mathbb{S}}_x + \hat{\mathbb{S}}_y)^2 - S(S+1)$$

$(\hat{\mathbb{S}}_x + \hat{\mathbb{S}}_y)^2$ → min. e.v. 0 non-deg.,
max e.v. $2S(2S+1)$ $(4S+1)$-fold deg.

$\hat{\mathbb{S}}_x \cdot \hat{\mathbb{S}}_y$ min. e.v. $-S(S+1)$ non-deg. ← singlet
max e.v. S^2 $(4S+1)$ fold deg.

$$-S(S+1) \le \hat{\mathbb{S}}_x \cdot \hat{\mathbb{S}}_y \le S^2$$

NOT symmetric

(classical vectors
$$-S^2 \le \mathbb{S}_x \cdot \mathbb{S}_y \le S^2$$)

symmetric

§ Ferromagnetic Heisenberg model (warmup)

connected lattice $(\Lambda, \mathcal{B})$

Λ: set of sites $x, y, \cdots$; $\mathcal{B}$: set of bonds $(x,y)=(y,x)$

$S=\frac{1}{2}, 1, \frac{3}{2}, \cdots$

Hamiltonian
$$\hat{H} = -\underbrace{\sum_{(x,y)\in\mathcal{B}} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y}$$
(spins want to align with each other)

(*) $[\hat{H}, \hat{S}^{(\alpha)}_{tot}]=0 \qquad \alpha=1,2,3$

<u>ground states</u>

$$\Phi_\uparrow := \bigotimes_{x\in\Lambda} \psi^S_x \qquad \uparrow\uparrow\uparrow\uparrow$$

Then $-\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \Phi_\uparrow = \underbrace{-S^2}_{\text{min. e.v. of } -\hat{\mathbf{S}}_x\cdot\hat{\mathbf{S}}_y} \Phi_\uparrow$

$\therefore$ $\Phi_\uparrow$ is a ground state $\hat{H}\Phi_\uparrow = \underbrace{-|\mathcal{B}|S^2}_{E_{GS}} \Phi_\uparrow$

other g.s.
$$\Phi_\ell := \frac{(\hat{S}^-_{tot})^\ell \Phi_\uparrow}{\|(S^-_{tot})^\ell \Phi_\uparrow\|}, \quad \ell=0,1,\cdots,2|\Lambda|S$$

$$\hat{H}\Phi_\ell = E_{GS}\Phi_\ell$$

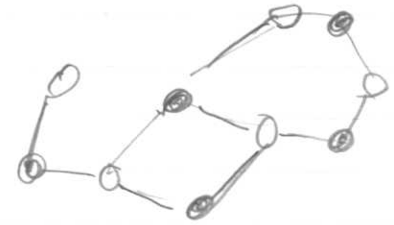
$2|\Lambda|S+1$ ground states

<u>QS-1</u> { (a) show (*)
(b) Show that Φ_ℓ is a g.s.

QS-2 — Show that these are the only g.s. → (hint on Day 3)

§ Antiferromagnetic Heisenberg model

(often called Heisenberg AF)



$(\Lambda, \mathcal{B})$ connected, bipartite.

$\Lambda = A \cup B$ $\quad (x,y) \in \mathcal{B} \Rightarrow x \in A, y \in B$ or $x \in B, y \in A$

$S = \frac{1}{2}, 1, \ldots$

Hamiltonian

spins want to point in the opposite direction.

$$\hat{H} = \sum_{(x,y)\in\mathcal{B}} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y \qquad \rightarrow [\hat{H}, \hat{S}^{(\alpha)}_{\text{tot}}] = 0$$

Néel state → ~~the~~ a ground state??

$$\Phi_{\text{Néel}} := \left(\bigotimes_{x\in A} \psi_x^{S}\right) \otimes \left(\bigotimes_{y\in B} \psi_y^{-S}\right)$$

Noting that $\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y = \hat{S}^{(3)}_x \hat{S}^{(3)}_y + \frac{1}{2}(\hat{S}^+_x \hat{S}^-_y + \hat{S}^-_x \hat{S}^+_y)$

$$(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y)(\psi_x^{S} \otimes \psi_y^{-S}) = \underbrace{-S^2 (\psi_x^{S} \otimes \psi_y^{-S})}_{\text{main if } S \gg 1 \text{ (classical)}} + S(\psi_x^{S-1} \otimes \psi_y^{-S+1})$$

$\Phi_{\text{Néel}}$ is not a g.s. (ulless $S = \infty$)

Theorem (Marshall 1955, Lieb-Mattis 1962)

Let $(\Lambda, \mathcal{B})$ be connected, bipartite with $|A| = |B|$.

Then the g.s. Φ_{GS} is unique and has $S_{tot} = 0$.

It can be expanded as

$$\Phi_{GS} = \sum_{\substack{\sigma \\ (\sum_{x\in\Lambda} \sigma_x = 0)}} C_\sigma \, (-1)^{\sum_{x\in A}(\sigma_x - S)} \, \Psi^\sigma$$

$$(-1)^{\sum_{x\in A}(\sigma_x - S)} \, \Psi^\sigma = \tilde{\Psi}^\sigma$$

with $C_\sigma > 0$.

proof Look for simultaneous eigenstates of $\hat{H}$, $\hat{S}^{(3)}_{tot}$, $(\hat{\mathbf{S}}_{tot})^2$

Suppose $\hat{H}\Phi = E\Phi$, $\hat{S}^{(3)}_{tot}\Phi = M\Phi$ with $M \neq 0$

↳ so $S_{tot} \geq |M|$

then $\hat{H}\underbrace{(\hat{S}^-_{tot})^M \Phi}_{\text{nonvanishing}} = (\hat{S}^-_{tot})^M \hat{H}\Phi$

$= E(\hat{S}^-_{tot})^M \Phi$.

We can find all the energy eigenvalues in the subspace with $\hat{S}^{(3)}_{tot} = 0$.

basis $\tilde{\Psi}^\sigma$ with $\sum_x \sigma_x = 0$

then (i) $\langle \tilde{\Psi}^\sigma, \hat{H}\tilde{\Psi}^{\sigma'} \rangle \in \mathbb{R}$

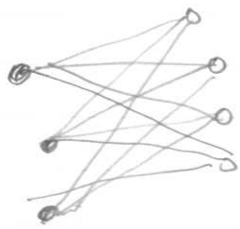
(ii) $\langle \tilde{\Psi}^\sigma, \hat{H}\tilde{\Psi}^{\sigma'} \rangle \leq 0$ if $\sigma \neq \sigma'$

(iii) $\forall \sigma, \sigma'$ with $\sum \sigma_x = \sum \sigma'_x = 0$ are connected via $\hat{H}$.

the PF theorem implies that

the $\underbrace{\text{g.s.}}_{\Phi_{GS}}$ within the subspace is unique, and $C_\sigma > 0$.

What is S_{tot} for Φ_{GS} ?



toy model on the same lattice

$$\hat{H}_{toy} = \left(\sum_{x\in A}{}' \hat{\mathbf{S}}_x\right)\cdot\left(\sum_{y\in B}{}' \hat{\mathbf{S}}_y\right)$$

$$= \frac{1}{2}\left\{\underbrace{(\hat{\mathbf{S}}_{tot})^2}_{=0} - \underbrace{(\hat{\mathbf{S}}_A)^2}_{=S|A|(S|A|+1)} - \underbrace{(\hat{\mathbf{S}}_B)^2}_{=S|B|(S|B|+1)}\right\}$$

We get the g.s. when

$$\therefore (\hat{\mathbf{S}}_{tot})^2\,\Phi_{toyGS} = 0$$

We also have $\Phi_{toyGS} = \sum'_{\boldsymbol{\sigma}\ (\Sigma\sigma_x=0)} C'(\boldsymbol{\sigma})\,\tilde{\Psi}^{\boldsymbol{\sigma}}$ with $C'(\boldsymbol{\sigma})>0$

$$\therefore \langle \Phi_{GS}, \Phi_{toyGS}\rangle \neq 0$$

$\Downarrow$ ← Φ_{GS} is an e.s. of $(\hat{\mathbf{S}}_{tot})^2$.

$$(\hat{\mathbf{S}}_{tot})^2\,\Phi_{GS} = 0$$

has $S_{tot}=0$ and hence

The G.S. is unique //

QS-3 extend the theorem to the case with $|A|\neq|B|$

QS-4 Verify (ii) in the previous page.

The nature of Φ_{GS} ? $\longrightarrow$ depends on $(\Lambda, \mathcal{B})$.

$d\geq 2$ today $\quad d=1$ day 2.

⟨LRO and SSB in quantum spin systems⟩

§ LRO in the ground state of the Heisenberg AF in $d \geq 2$

$L \times \cdots \times L$ d-dim. hypercubic lattice

$$(\Lambda_L, \mathcal{B}_L)$$

$\Lambda_L = A \cup B$ with → bipartite!

$$A = \{x = (x_1, \dots, x_d) \in \Lambda_L \mid \textstyle\sum_i x_i \text{ even}\}$$

$$B = \{x = (x_1, \dots, x_d) \in \Lambda_L \mid \textstyle\sum x_i \text{ odd}\}$$

Hamiltonian

$$\hat{H} := \sum_{(x,y) \in \mathcal{B}_L} \hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y$$

Symmetry
global spin rotation $\alpha = 1, 2, 3$ — SU(2) (rather than SO(3))

$$\hat{U} = \exp\left[i\theta \sum_{x \in \Lambda_L} \hat{S}_x^{(\alpha)}\right]$$

AF order parameter

$$\hat{\mathcal{O}}^{(\alpha)} := \sum_{x \in \Lambda_L} (-1)^x \hat{S}_x^{(\alpha)} \qquad \alpha = 1, 2, 3$$

$$(-1)^x = \begin{cases} 1 & x \in A \\ -1 & x \in B \end{cases}$$

<u>Theorem</u> $(d \geq 3, \forall S)$, OR $(d \geq 2, S \geq 1)$, then $\exists q_0 > 0$ (depends on d, S but not on L) s.t.

$$\frac{1}{L^{2d}} \langle \Phi_{GS}, (\hat{\mathcal{O}}^{(\alpha)})^2 \Phi_{GS} \rangle \geq q_0 \text{ for } \forall L$$

$\alpha = 1, 2, 3$

(proof uses reflection positivity — very difficult — due to Dyson-Lieb-Simon 1978
Neves-Perez 1986, Kennedy-Lieb-Shastry 1988, Kubo-Kishi 1988, ...)

Thus.

$$(-1)^{x-y} \langle \Phi_{GS}, \hat{S}_x \cdot \hat{S}_y \Phi_{GS} \rangle \gtrsim 3 q_0$$

for $\forall x, y$

long-range AF order (or Néel order)

But the uniqueness implies

$$\langle \Phi_{GS}, \hat{O}^{(\alpha)} \Phi_{GS} \rangle = 0 \quad \text{for } \alpha = 1,2,3$$

NO SSB

"LRO without SSB" is common in the g.s. of quantum many-body systems where the Hamiltonian and the order parameter do not commute.

quantum field theory
superconductivity
Bose-Einstein cond.

the simplest example

$\Downarrow$

DATE

§ Ising model under transverse magnetic field in $d=1$

$$S=\frac{1}{2},\quad \hat{H} = -\sum_{x=1}^{L} \hat{S}_x^{(3)}\hat{S}_{x+1}^{(3)} - \delta\sum_{x=1}^{L}\hat{S}_x^{(1)} \quad (\delta \geq 0)$$

$\delta = 0$ Ising ferro

two g.s. $\Phi_\uparrow = \bigotimes_{x=1}^{L} \psi_x^\uparrow$, $\Phi_\downarrow = \bigotimes_{x=1}^{L} \psi_x^\downarrow$

$$E_{GS} = -\frac{L}{4}$$

kinks

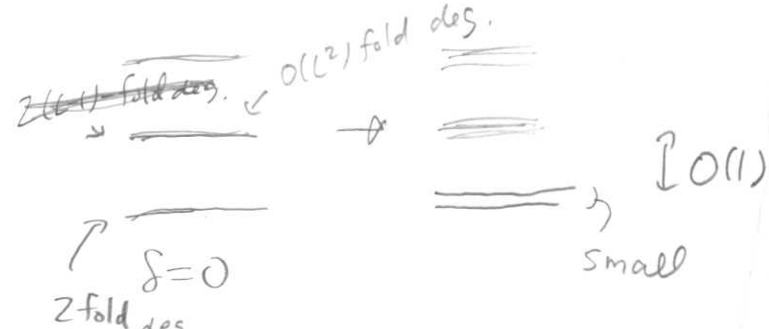
1st excited states $\uparrow\uparrow\uparrow\downarrow\downarrow\cancel{\uparrow}\cancel{\uparrow}$ $E_{1st} = E_{GS} + 1$

$0 < \delta \ll 1$ unique g.s.

$$\Phi_{GS} \simeq \frac{1}{\sqrt{2}}(\Phi_\uparrow + \Phi_\downarrow)$$

CATS

$$\Phi_{1st} \simeq \frac{1}{\sqrt{2}}(\Phi_\uparrow - \Phi_\downarrow)$$



low-lying excitated state $E_{1st} - E_{GS} \propto \delta^L$

Symmetry π-rotation around the 1-axis $\hat{U} = e^{i\pi\sum_{x=1}^{L}\hat{S}_x^{(1)}}$

$$\hat{U}^{-1}\hat{S}_x^{(3)}\hat{U} = -\hat{S}_x^{(3)}$$

order parameter $\mathcal{O} = \hat{S}_{tot}^{(3)} = \sum_{x=1}^{L}\hat{S}_x^{(3)}$

$$\mathcal{O}\,\Phi_\uparrow = \frac{L}{2}\Phi_\uparrow,\quad \mathcal{O}\,\Phi_\downarrow = -\frac{L}{2}\Phi_\downarrow$$

$$\begin{cases} \langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle \simeq \frac{L^2}{4} & \text{LRO} \\ \langle \Phi_{GS}, \hat{\Theta}\, \Phi_{GS} \rangle = 0 & \text{without SSB} \end{cases}$$

from the uniqueness of the g.s.

Φ_{GS} : exact g.s. for finite L, but <u>unphysical</u>

$$\sqrt{\langle \Phi_{GS}, \left(\frac{\hat{\Theta}}{L}\right)^2 \Phi_{GS} \rangle - \langle \Phi_{GS}, \frac{\hat{\Theta}}{L} \Phi_{GS} \rangle^2} \simeq \frac{1}{2}$$

$\left(\frac{\hat{\Theta}}{L}\right)$ fluctuates!!

physically natural "g.s." are $\Phi_\uparrow$ and $\Phi_\downarrow$

$$\begin{cases} \langle \Phi_\uparrow, \hat{\Theta}^2 \Phi_\uparrow \rangle = \frac{L^2}{4} & \text{LRO} \\ & + \\ \langle \Phi_\uparrow, \hat{\Theta}\, \Phi_\uparrow \rangle = \frac{L}{2} & \text{SSB} \end{cases}$$

$\frac{\hat{\Theta}}{L}$ does not fluctuate!

$$\sqrt{\langle \Phi_\uparrow, \left(\frac{\hat{\Theta}}{L}\right)^2 \Phi_\uparrow \rangle - \langle \Phi_\uparrow, \frac{\hat{\Theta}}{L} \Phi_\uparrow \rangle^2} = 0$$

<u>Observations</u>

① $$\begin{cases} \Phi_\uparrow \simeq \frac{1}{\sqrt{2}}(\Phi_{GS} + \Phi_{1st}) \\ \Phi_\downarrow \simeq \frac{1}{\sqrt{2}}(\Phi_{GS} - \Phi_{1st}) \end{cases}$$

physical "g.s." are linear combinations of the exact g.s. and the low-lying excited state

② $$\hat{\Theta}\Phi_{GS} \simeq \frac{1}{\sqrt{2}}\Big(\overbrace{\hat{\Theta}\Phi_\uparrow}^{\frac{L}{2}\Phi_\uparrow} + \overbrace{\hat{\Theta}\Phi_\downarrow}^{-\frac{L}{2}\Phi_\downarrow}\Big) \simeq \text{const.}\, \Phi_{1st}$$

§ From LRO to SSB   Kaplan - Horsch - von der Linden.

consider
- Ising under trans. field
- Heisenberg AF on $\Lambda_L \subset \mathbb{Z}^d$

  or

  more general models on Λ_L

$\hat{\Theta}^\dagger = \hat{\Theta}$ is crucial

$$\hat{\Theta} = \begin{cases} \hat{S}^{(3)}_{tot} & \text{Ising} \\ \hat{\Theta}^{(\alpha)} = \sum_x (-1)^x \hat{S}^{(\alpha)}_x & \text{Heisenberg AF} \end{cases}$$

assume $\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle \geq q_0 L^{2d}$ LRO

$\langle \Phi_{GS}, \hat{\Theta}^n \Phi_{GS} \rangle = 0 \quad (n = 1, 3, \ )$ NO SSB

<u>construction of low-lying excited state</u>   Horsch - von der Linden 1988

trial state $\Gamma = \dfrac{\hat{\Theta}\Phi_{GS}}{\|\hat{\Theta}\Phi_{GS}\|}$, $\langle \Phi_{GS}, \Gamma \rangle = 0$

$$\langle \Gamma, \hat{H}\Gamma \rangle - E_{GS}$$

$$= \frac{\langle \Phi_{GS}, \hat{\Theta}\hat{H}\hat{\Theta}\Phi_{GS} \rangle - \frac{1}{2}\langle \Phi_{GS}, \hat{\Theta}^2 \hat{H}\Phi_{GS} \rangle - \frac{1}{2}\langle \Phi_{GS}, \hat{H}\hat{\Theta}^2 \Phi_{GS} \rangle}{\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}$$

$$= \frac{\langle \Phi_{GS}, [\hat{\Theta}, [\hat{H}, \hat{\Theta}]] \Phi_{GS} \rangle}{2\langle \Phi_{GS}, \hat{\Theta}^2 \Phi_{GS} \rangle}$$

now $[\hat{H},\hat{\theta}] = \sum_x \bigcirc_x$ ← local around x

$[\hat{\theta},[\hat{H},\hat{\theta}]] = \sum_x \bigcirc_x$

$\therefore \quad \|[\hat{\theta},[\hat{H},\hat{\theta}]]\| \leq \text{const}\, L^d$

$\therefore$

$$0 \leq \langle\Gamma,\hat{H}\,\Gamma\rangle - E_{GS} \leq \frac{\text{const}\, L^d}{2 q_0 L^{2d}} = C L^{-d}$$

Theorem $\quad E_{1st} \leq E_{GS} + C L^{-d}$

(LRO without SSB $\longrightarrow$ $\exists$ low-lying excited state)

Low-lying states with SSB

$\Xi = \frac{1}{\sqrt{2}}(\Phi_{GS}+\Gamma)$, $\quad \langle\Xi,\hat{H}\,\Xi\rangle \leq E_{GS} + \frac{C}{2} L^{-d}$

low-lying state

$$\langle\Xi,\hat{\theta}\,\Xi\rangle = \frac{1}{2}\left\langle\left(\Phi_{GS}+\frac{\hat{\theta}\,\Phi_{GS}}{\|\hat{\theta}\,\Phi_{GS}\|}\right),\left(\hat{\theta}\,\Phi_{GS}+\frac{\hat{\theta}^2\,\Phi_{GS}}{\|\hat{\theta}\,\Phi_{GS}\|}\right)\right\rangle$$

$$= \frac{\langle\Phi_{GS},\hat{\theta}^2\,\Phi_{GS}\rangle}{\|\hat{\theta}\,\Phi_{GS}\|} = \sqrt{\langle\Phi_{GS},\hat{\theta}^2\,\Phi_{GS}\rangle}$$

$$\geq \sqrt{q_0}\, L^d$$

Ξ is a low-lying state with SSB → physical "g.s."

so is $\frac{1}{\sqrt{2}}(\Phi_{GS}-\Gamma)$

DATE

SSB under "infinitessimally small external field"

Hamiltonian with (staggered) magnetic field

$$\hat{H}_h = \hat{H} - h\hat{\mathcal{O}}, \qquad h>0$$

$\Phi_{GS,h}$ the GS of H_h

Obviously

$$\langle \Xi, \underbrace{\hat{H}_h}_{\hat{H}-h\hat{\mathcal{O}}} \Xi\rangle \geq \langle \Phi_{GS,h}, \underbrace{\hat{H}_h}_{\hat{H}-h\hat{\mathcal{O}}} \Phi_{GS,h}\rangle$$

devide by hL^d

$$\frac{1}{L^d}\langle \Phi_{GS,h}, \hat{\mathcal{O}}\Phi_{GS,h}\rangle \geq \frac{1}{L^d}\langle \Xi, \hat{\mathcal{O}}\Xi\rangle$$

$$+\frac{1}{hL^d}\{\langle \Phi_{GS,h}, \hat{H}\Phi_{GS,h}\rangle - \langle \Xi, \hat{H}\Xi\rangle\}$$

$$\geq \sqrt{q_0} + \frac{1}{hL^d}\{\underbrace{E_{GS} - \langle \Xi, \hat{H}\Xi\rangle}_{\text{"}O(L^{-d})\text{"}}\}$$

the gs energy with $h=0$

Theorem (Kaplan-Horsch-von der Linden, 1989)

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\frac{1}{L^d}\langle \Phi_{GS,h}, \hat{\mathcal{O}}\Phi_{GS,h}\rangle \geq \sqrt{q_0}$$

$$\boxed{\underset{(h=0)}{\text{LRO}} \rightarrow \underset{\text{LRO}}{\overset{}{\text{SSB}}\ +}}\ (h\downarrow 0)$$

for quite general quantum many-body systems

NOT YET THE WHOLE STORY!

§ From LRO to SSB   Koma-Tasaki theorems (1994)
and improvements (Tasaki 2015) unpublished

systems with a continuous symmetry
↓
infinitely many "g.s." with SSB
many low-lying states? → Yes

Heisenberg AF on Λ_L (or other lattice models with contin. sym.)
↳ $\boxed{SU(2)\text{ symmetry}}$

order par.

$$\hat{O}^{(\alpha)} := \sum_{x\in\Lambda_L} (-1)^x \hat{S}_x^{(\alpha)}, \quad \hat{O}^{\pm} := \sum_{x\in\Lambda_L} (-1)^x \hat{S}_x^{\pm}$$

Φ_{GS} unique finite volume g.s.
$$\begin{cases} \hat{H}\Phi_{GS} = E_{GS}\Phi_{GS} \\ \hat{S}^{(3)}_{tot}\Phi_{GS} = 0 \end{cases}$$

- exhibits LRO without SSB

$$\begin{cases} \dfrac{1}{L^{2d}}\langle \Phi_{GS}, (\hat{O}^{(\alpha)})^2 \Phi_{GS}\rangle \geq q_0 \quad \text{for } \forall L \\ \dfrac{1}{L^{2d}}\langle \Phi_{GS}, \hat{O}^{(\alpha)} \Phi_{GS}\rangle = 0 \end{cases}$$

For $M = 1, 2, \cdots$ let

$$\Gamma_M := \frac{(\hat{\Theta}^+)^M \Phi_{GS}}{\|(\hat{\Theta}^+)^M \Phi_{GS}\|}, \quad \Gamma_{-M} := \frac{(\hat{\Theta}^-)^M \Phi_{GS}}{\|(\hat{\Theta}^-)^M \Phi_{GS}\|}$$

Theorem For $\forall M$ s.t. $|M| \leq \text{const.}\, L^{d/2}$,

$$\langle \Gamma_M, \hat{H}\Gamma_M \rangle \leq E_{GS} + \text{const.} \frac{M^2}{L^d}$$

(proof : NOT easy)

Since $\hat{S}^{(3)}_{tot}\Gamma_M = M\Gamma_M$,

$\exists\, \Phi_M$ s.t. $\begin{cases} \hat{S}^{(3)}_{tot}\Phi_M = M\Phi_M \\ \hat{H}\Phi_M = E_M\Phi_M \text{ with } E_{GS} < E_M \leq E_{GS} + \text{const} \dfrac{M^2}{L^d} \end{cases}$

$\exists$ ever increasing series of low-lying excited states.

$\downarrow$

well-known in numerical community Kikuchi ... 1990

"Anderson's Tower"

$d=3$ excitation energy $\sim \dfrac{1}{L^3}$ (spin wave exc. $\sim \dfrac{1}{L^2}$)

low-lying state(s) with full SSB

$$\textcircled{H}_L := \frac{1}{\sqrt{2M_{max}(L)+1}}\left\{\Phi_{GS} + \sum_{M=1}^{M_{max}(L)} (\Gamma_M + \Gamma_{-M})\right\}$$

with $M_{max}(L) \uparrow \infty$ as $L \uparrow \infty$ not too rapidly

$$m^* := \lim_{k\uparrow\infty} \lim_{L\uparrow\infty} \left\langle \Phi_{GS}, \left(\frac{\hat{O}^{(\alpha)}}{L^d}\right)^{2k} \Phi_{GS} \right\rangle^{1/(2k)}$$

$(\alpha = 1,2,3)$ $\left(\sqrt{2}q_0 \text{ for } U(1) \text{ symmetric models}\right)$

Theorem $\langle \textcircled{H}_L, \hat{O}^{(\alpha)} \textcircled{H}_L \rangle = 0 \quad \alpha = 2,3$

$$\lim_{L\uparrow\infty} \left\langle \textcircled{H}_L, \frac{\hat{O}^{(1)}}{L^d} \textcircled{H}_L \right\rangle = m^* \geq \sqrt{3}q_0 \qquad \boxed{\text{SSB}}$$

Néel order

$$\lim_{L\uparrow\infty} \left\langle \textcircled{H}_L, \left(\frac{\hat{O}^{(1)}}{L^d}\right)^2 \textcircled{H}_L \right\rangle = (m^*)^2 \qquad \boxed{\text{LRO}}$$

$\rightarrow$ $\hat{O}^{(1)}/L^d$ does not fluctuate as $L \uparrow \infty$

Physical "g.s." with LRO and SSB are linear combinations of ever increasing number of low lying states!!

Theorem Let $\Phi_{GS,h}$ be the g.s. of $\hat{H} - h\hat{O}^{(1)}$

$$\lim_{h\downarrow 0} \lim_{L\uparrow\infty} \left\langle \Phi_{GS,h}, \frac{\hat{O}^{(1)}}{L^d} \Phi_{GS,h} \right\rangle = m^* \geq \sqrt{3}q_0$$

Why $\sqrt{3q_0}$?

$$\frac{(\hat{O}^{(1)}, \hat{O}^{(2)}, \hat{O}^{(3)})}{L^d} \xrightarrow[L\to\infty]{} \vec{O} \leftarrow \text{classical vector}$$

$$\langle |\vec{O}|^2 \rangle_{GS} = \langle (O^{(1)})^2 \rangle + \langle (O^{(2)})^2 \rangle + \langle (O^{(3)})^2 \rangle$$
$$= 3q_0$$

$$|\vec{O}|^2 = 3q_0$$

§ Ground states of infinite systems

Heisenberg AF on Λ_L, assume $^\exists$LRO in Φ_{GS} the unique g.s.

algebra of operators

$$\tilde{\mathfrak{A}} = \{\text{polynomials of } \hat{S}_x^{(\alpha)},\ x \in \mathbb{Z}^d,\ \alpha=1,2,3\}$$

\mathfrak{A}

$$\omega_0(\hat{A}) := \lim_{L\uparrow\infty} \langle \Phi_{GS}, \hat{A}\, \Phi_{GS} \rangle$$

Ω : solid angle. $\hat{U}_\Omega$ a suitable rotation $(1,0,0) \to \Omega$

$$\omega_\Omega(\hat{A}) := \lim_{L\uparrow\infty} \langle \hat{U}_\Omega \Theta_L,\ \hat{A}\, \hat{U}_\Omega \Theta_L \rangle$$

<u>Theorem</u> (Koma-Tasaki)

$\omega_0(\cdot)$ and $\omega_\Omega(\cdot)$ are g.s. infinitely many g.s. !

$$\left(\begin{array}{l} \text{i.e., for } ^\forall(x,y) \text{ s.t. } |x-y|=1 \\ \quad \omega_0(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y) = \omega_\Omega(\hat{\mathbf{S}}_x \cdot \hat{\mathbf{S}}_y) = \epsilon_{GS} := \lim_{L\uparrow\infty} \dfrac{E_{GS,L}}{|\mathcal{B}_L|} \end{array} \right)$$

$$\omega_0(\hat{S}_x^{(\alpha)}) = 0 \qquad \alpha = 1,2,3$$

$$(-1)^x\, \omega_\Omega(\Omega \cdot \hat{\mathbf{S}}_x) = m^* \geq \sqrt{3 q_0}$$

$$\omega_\Omega(\mathbf{v} \cdot \hat{\mathbf{S}}_x) = 0 \quad \text{if } \mathbf{v}\cdot\Omega = 0$$

and

$$\omega_0(\cdot) = \frac{1}{4\pi} \int d\Omega\, \omega_\Omega(\cdot)$$

//

$\omega_0(\cdot)$ is ~~not~~ unphysical ergodic $\left(\frac{\hat{O}_L^{(\alpha)}}{L^d}\right)$ has big fluctuation

Conjecture

$\omega_\Omega(\cdot)$ is ergodic (physical state)

$\left(\begin{array}{l}\forall \text{macroscopic quantities has small} \\ \text{fluctuation in } \omega_\Omega(\cdot)\end{array}\right)$

Then

$$\omega_0(\cdot) = \overbrace{\frac{1}{4\pi}\int d\Omega\ \omega_\Omega(\cdot)}^{\text{mathomatically natural decomposition into ergodic states}}$$

$\omega_0(\cdot)$ ↓ unphysical g.s. $\boxed{\text{LRO without SSB}}$

(obtained from the unique g.s. Φ_{GS}).

$\omega_\Omega(\cdot)$ ↓ physical g.s. with Néel order.

in reality one of $\omega_\Omega(\cdot)$ is selected (by some reasons).

$\boxed{\text{SSB}}$

how??

§ thermal equilibrium (remarks)

Heisenberg model on Λ_L

$d=1,2$ no LRO or SSB if $T \gneq 0$. (Hohenberg, 1967 Mermin-Wagner) 1966

ferro or AF

$d \geq 3$ AF LRO at suff. low temperatures

(Dyson-Lieb-Simon 1978
Kennedy-Lieb-Shastry 1988)

BEC of spin waves

SSB (Koma-Tasaki 1993)

↓

$\hat{H}$ and $\hat{\Theta}$ "almost commute" for large L

↓

extension of the Griffiths' theorem

"physics" may not be very different from classical situation

no results for Heisenberg ferro!

⟨LRO and SSB associated with Bose-Einstein condensation⟩

§ Hard core bosons on a (optical) lattice.

$L \times \cdots \times L$ d-dim. hypercubic lattice $(\Lambda_L, \mathcal{B}_L)$

$$\begin{cases} \hat{a}_x & \text{annihilation operator of a boson at } x \in \Lambda_L \\ \hat{a}^\dagger_x & \text{creation} \quad \quad \quad '' \end{cases}$$

$$[\hat{a}_x, \hat{a}^\dagger_y] = \delta_{x,y}$$

Φ_{vac} unique state s.t. $\hat{a}_x \Phi_{vac} = 0$ for $\forall x$

$\longrightarrow$ state with no bosons

Hilbert space of N boson system is spanned by

$$\underbrace{\hat{a}^\dagger_{x_1} \hat{a}^\dagger_{x_2} \cdots \hat{a}^\dagger_{x_N}}_{N} \Phi_{vac}$$

with any $x_1, \cdots, x_N \in \Lambda_L$ s.t. $\underbrace{x_i \neq x_j \text{ if } i \neq j}_{\text{hard core}}$

fix $\rho = \dfrac{N}{L^d}$ and change L, N.

the simplest (standard) Hamiltonian

$$\hat{H} = -t \sum_{(x,y) \in \mathcal{B}_L} \underbrace{(\hat{a}^\dagger_x \hat{a}_y + \hat{a}^\dagger_y \hat{a}_x)}_{\text{hopping}} \quad (t \geq 0)$$

§ off-diagonal LRO

relevant symmetry for BEC

U(1) gauge symmetry $\hat{U}(\theta) = e^{i\theta\hat{N}}$

$$\hat{N} = \sum_{x\in\Lambda_L} \hat{n}_x, \quad \hat{n}_x = \hat{a}_x^\dagger \hat{a}_x$$

order parameters

$$\hat{\mathcal{O}}^+ = \sum_x \hat{a}_x^\dagger, \quad \hat{\mathcal{O}}^- = \sum_x \hat{a}_x$$

or $\hat{\mathcal{O}}^{(1)} = \frac{1}{2}\{\hat{\mathcal{O}}^+ + \hat{\mathcal{O}}^-\}, \quad \hat{\mathcal{O}}^{(2)} = \frac{1}{2i}\{\hat{\mathcal{O}}^+ - \hat{\mathcal{O}}^-\}$

If $d \geq 2$, it is expected that there is BEC for a wide range of ρ. (rigorous only for $\rho = 1/2$)

$$\left\langle \Phi_{GS}, \left(\frac{\hat{\mathcal{O}}^{(\alpha)}}{L^d}\right)^2 \Phi_{GS} \right\rangle \geq q_0 > 0 \quad \text{for } ^\forall L \quad (\alpha = 1, 2)$$

$$\Downarrow$$

$$\langle \Phi_{GS}, \hat{a}_x^\dagger \hat{a}_y \Phi_{GS} \rangle \gtrsim 2q_0 \quad \text{for } ^\forall x, y$$

off-diagonal LRO

BUT clearly

N bosons

$$\langle \Phi_{GS}, \hat{\mathcal{O}}^{(\alpha)} \Phi_{GS} \rangle = 0 \quad (\alpha = 1, 2)$$

(Kubo-Kishi 1988, Kennedy-Lieb-Shastry 1988)

§ "g.s." with SSB

Hilbert space with any number of bosons

$$\hat{H} = -t \sum_{(x,y)}{}^{'} (\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger a_x) - \mu \hat{N}$$

chose μ so that the g.s. Φ_{GS} has ~~given~~ the desired ρ.

Koma-Tasaki construction of low-lying states (Tasaki 2015)

$$\Theta_{L,\varphi} := \frac{1}{\sqrt{2M_{max}(L)+1}} \left\{ \Phi_{GS} + \sum_{M=1}^{M_{max}(L)} \left(\frac{e^{-i\varphi M} (\hat{O}^+)^M \Phi_{GS}}{\|(\hat{O}^+)^M \Phi_{GS}\|} + \frac{e^{i\varphi M} (\hat{O}^-)^M \Phi_{GS}}{\|(\hat{O}^-)^M \Phi_{GS}\|} \right) \right\}$$

$$\lim_{L\uparrow\infty} \left\langle \Theta_{L,\varphi}, \frac{\hat{O}^\pm}{L^d} \Theta_{L,\varphi} \right\rangle = m^* e^{\pm i\varphi}$$

$$\lim_{L\uparrow\infty} \left\langle \Theta_{L,\varphi}, \frac{\hat{O}^{(\alpha)}}{L^d} \Theta_{L,\varphi} \right\rangle = \begin{cases} m^* \cos\varphi & (\alpha=1) \\ m^* \sin\varphi & (\alpha=2) \end{cases}$$

$$m^* \geq \sqrt{2q_0}$$

$$\lim_{L\uparrow\infty} \left\langle \Theta_{L,\varphi}, \left(\frac{\hat{O}^{(\alpha)}}{L^d}\right)^2 \Theta_{L,\varphi} \right\rangle = \begin{cases} (m^* \cos\varphi)^2 & (\alpha=1) \\ (m^* \sin\varphi)^2 & (\alpha=2) \end{cases}$$

LRO and SSB !

infinite volume g.s.

$$\omega_0(\cdot) = \lim_{L\uparrow\infty} \langle \Phi_{GS}, (\cdot)\, \Phi_{GS} \rangle$$

$$\omega_\varphi(\cdot) = \lim_{L\uparrow\infty} \langle \Theta_{L,\varphi}, (\cdot)\, \Theta_{L,\varphi} \rangle$$

$$\underbrace{\omega_0(\cdot)}_{\substack{\text{ODLRO} \\ \text{without } U(1)\text{ SSB}}} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \underbrace{\omega_\varphi(\cdot)}_{\text{ODLRO} + U(1)\text{SSB}}$$

BUT superposition of states with different boson numbers is meaningless.

$$\underbrace{\Phi_N + \Phi_{N+1}}_{\text{NOT ALLOWED}} \longrightarrow \underbrace{\Phi_N \otimes \Xi_{\mathcal{N}-N} + \Phi_{N-1} \otimes \Xi_{\mathcal{N}-N+1}}_{\text{ALLOWED}}$$

$$\underbrace{\omega_0(\cdot)}_{\substack{\text{physical} \\ \text{g.s.} \\ \text{realized in} \\ \text{an optical} \\ \text{lattice}}} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \underbrace{\omega_\varphi(\cdot)}_{\substack{\text{fictitious states} \\ \text{which are "natural" from} \\ \text{theoretical point of view}}}$$

(→ realistic for photons!)

→ BCS theory

$\boxed{\text{the same picture for superconductivity}}$

DATE

§ Physical "SSB" in a coupled system.

two identical lattices Λ_L and Λ_L'

$$\hat{H}_\varepsilon = \hat{H}\otimes\hat{\mathbb{1}} + \hat{\mathbb{1}}\otimes\hat{H} - \varepsilon\{e^{-i\varphi}\hat{O}^-\hat{O}'^+ + e^{i\varphi}\hat{O}^+\hat{O}'^-\}$$

$$\text{''} \sum_{\substack{x\in\Lambda_L\\ y\in\Lambda_L'}} \left(e^{-i\varphi}\hat{a}_x\hat{a}^\dagger_y + e^{i\varphi}\hat{a}^\dagger_x\hat{a}_y\right)$$

Hermitian, number conserving (gauge invariant)

$\Phi^{(\varphi)}_{GS,\varepsilon}$: the g.s. in the constant number Hilbert space with $2N$ bosons.

Theorem

$$\lim_{\varepsilon\downarrow 0}\lim_{L\uparrow\infty}\frac{1}{L^{2d}}\left\langle\Phi^{(\varphi)}_{GS,\varepsilon},\left(\sum_{x\in\Lambda_L}\hat{a}^\dagger_x\right)\left(\sum_{y\in\Lambda_L'}\hat{a}_y\right)\Phi^{(\varphi)}_{GS,\varepsilon}\right\rangle = (m^*)^2 e^{-i\varphi}$$

$(m^*)^2 \geq 2q_0$.

$\boxed{\text{SSB for relative phase}}$

trial state

gauge invariant

$$\hat{\Theta}^{(\varphi)}_L = \frac{1}{2\pi}\int_0^{2\pi}d\theta\left(\Theta_{L,\theta}\otimes\Theta_{L,\theta+\varphi}\right)$$

$$= \frac{1}{2M_{max}(L)+1}\sum_{M=-M_{max}(L)}^{M_{max}(L)} e^{iM\varphi}\underbrace{\frac{(\hat{O}^+)^M\Phi_{GS}}{\|(\hat{O}^-)^M\Phi_{GS}\|}\otimes\frac{(\hat{O}^-)^M\Phi_{GS}}{\|(\hat{O}^-)^M\Phi_{GS}\|}}_{}$$

↑ N boson state!