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Symbols and notations

General

- $A := B$ or $B =: A$ defines A in terms of B.
- $I[\cdot]$ denotes the indicator function defined by $I[\text{true}] = 1$, $I[\text{false}] = 0$.
- The number of elements in a set S is denoted as $|S|$.
- $\sum_{\substack{j=0 \\ (j \neq i)}}^{\infty} f(j)$, for example, means that there is an extra condition $j \neq i$ in the sum.

Lattice

- Most generally a lattice structure is specified as $(\Lambda, \mathcal{B})$, where the set of sites Λ is a finite set, and the set of bonds $\mathcal{B}$ is a subset of $\Lambda \times \Lambda$ such that $(x, x) \notin \mathcal{B}$. We always identify (x, y) with (y, x).
- A lattice $(\Lambda, \mathcal{B})$ is *connected* if for any $x \neq y \in \Lambda$, there is a sequence $x_0, x_1, \ldots, x_n \in \Lambda$ such that $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in \mathcal{B}$ for $i = 0, 1, \ldots, n-1$.
- A lattice $(\Lambda, \mathcal{B})$ is *bipartite* if Λ is decomposed as $\Lambda = A \cup B$ where $A \cap B = \emptyset$, and $(x, y) \in \mathcal{B}$ implies $x \in A$, $y \in B$ or $x \in B$, $y \in A$.
- $d = 1, 2, \ldots$ dimension
- $\mathbb{Z}^d = \left\{ x = (x_1, \ldots, x_d) \,\middle|\, x_j \in \mathbb{Z} \right\}$ infinite d-dimensional hypercubic lattice
- $\Lambda_L = \left\{ x = (x_1, \ldots, x_d) \,\middle|\, x_j \in \mathbb{Z}, -L/2 < x_j \leq L/2 \right\} \subset \mathbb{Z}^d$ finite d-dimensional hypercubic lattice. L is even. For $d = 1$, I sometimes use $\Lambda_L = \{1, 2, \ldots, L\}$.
- $\mathcal{B}_L = \left\{ (x, y) \,\middle|\, x, y \in \Lambda_L, |x - y| = 1 \right\}$ the set of bonds. We use periodic boundary conditions, and include pairs of sites at the opposite ends of Λ_L into $\mathcal{B}_L$. We always identify (x, y) with (y, x).

Ising model

- $\sigma_x \in \{-1, +1\}$ spin variable at site x
- $\boldsymbol{\sigma} = (\sigma_x)_{x \in \Lambda_L} \in \{-1, +1\}^{\Lambda_L}$ spin configuration on Λ_L
- $H = -\sum_{(x,y) \in \mathcal{B}_L} \sigma_x \sigma_y$ Hamiltonian without external magnetic field.

- $Z_L(\beta) = \sum_{\boldsymbol{\sigma}} e^{-\beta H}$ partition function
- $\langle F \rangle_{\beta,L} = \{Z_L(\beta)\}^{-1} \sum_{\boldsymbol{\sigma}} F\, e^{-\beta H}$ expectation value at the inverse temperature β and vanishing external magnetic field.
- $\mathcal{O} = \sum_{x \in \Lambda_L} \sigma_x$ order parameter or total magnetization
- $H_h = H - h\mathcal{O}$ Hamiltonian under magnetic field $h \geq 0$
- $\langle F \rangle_{\beta,h,L} = \dfrac{\sum_{\boldsymbol{\sigma}} F\, e^{-\beta H_h}}{\sum_{\boldsymbol{\sigma}} e^{-\beta H_h}}$ expectation value at the inverse temperature β and under external magnetic field h.

Classical Heisenberg model

- $S_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)})$ with $\sum_{\alpha=1}^{3} \{S_x^{(\alpha)}\}^2 = 1$ spin variable at site x
- $dS_x = dS_x^{(1)} dS_x^{(2)} dS_x^{(3)} \delta(\sum_{\alpha=1}^{3} \{S_x^{(\alpha)}\}^2 - 1)$

Quantum spin systems

- $S = 1/2, 1, 3/2, 2, \ldots$ the "magnitude" of spin, which is a fixed constant
- $\mathcal{H}_x = \mathbb{C}^{2S+1}$ the Hilbert space at site $x \in \Lambda$
- $\hat{\boldsymbol{S}}_x = (\hat{S}_x^{(1)}, \hat{S}_x^{(2)}, \hat{S}_x^{(3)})$ spin operator acting on $\mathcal{H}_x$, which satisfies $[\hat{S}_x^{(\alpha)}, \hat{S}_x^{(\beta)}] = i \sum_\gamma \epsilon_{\alpha,\beta,\gamma} \hat{S}_x^{(\gamma)}$ and $(\hat{\boldsymbol{S}}_x)^2 = S(S+1)$
- $\hat{S}_x^{\pm} := \hat{S}_x^{(1)} \pm i \hat{S}_x^{(2)}$
- ψ_x^σ with $\sigma = -S, -S+1, \ldots, S$ denote the standard basis states of $\mathcal{H}_x$, which satisfies $\hat{S}_x^{(3)} \psi_x^\sigma = \sigma\, \psi_x^\sigma$ and $\hat{S}_x^{\pm} \psi_x^\sigma = \sqrt{S(S+1) - \sigma(\sigma \pm 1)}\, \psi_x^{\sigma \pm 1}$
- $\mathcal{H} := \bigotimes_{x \in \Lambda} \mathcal{H}_x$ the whole Hilbert space
- $\Psi^{\boldsymbol{\sigma}} := \bigotimes_{x \in \Lambda} \psi_x^{\sigma_x}$ with $\boldsymbol{\sigma} = (\sigma_x)_{x \in \Lambda}$ are the basis states
- $\hat{\boldsymbol{S}}_{\text{tot}} := \sum_{x \in \Lambda} \hat{\boldsymbol{S}}_x$, $\hat{S}_{\text{tot}}^{\pm} := \sum_{x \in \Lambda} \hat{S}_x^{\pm}$, $\hat{S}_{\text{tot}}^{(3)} := \sum_{x \in \Lambda} \hat{S}_x^{(3)}$, and the eigenvalues of $(\hat{\boldsymbol{S}}_{\text{tot}})^2$ are denoted as $S_{\text{tot}}(S_{\text{tot}}+1)$ with $S_{\text{tot}} = 0, 1, 2, \ldots, NS$ or $S_{\text{tot}} = 1/2, 3/2, \ldots, NS$

- $\hat{H} = \sum_{(x,y)\in\mathcal{B}_L} \hat{\boldsymbol{S}}_x \cdot \hat{\boldsymbol{S}}_y$ Hamiltonian for the Heisenberg antiferromagnet

- $\hat{\mathcal{O}}^{(\alpha)} = \sum_{x\in\Lambda_L} (-1)^x \, \hat{S}_x^{(\alpha)}$ with $\alpha = 1, 2, 3$ antiferromagnetic order parameter

- $\hat{\mathcal{O}}^{\pm} = \hat{\mathcal{O}}^{(1)} \pm i\hat{\mathcal{O}}^{(2)}$ the corresponding raising and lowering operators

- $\omega(\cdot)$ state of an infinite (quantum) system

Bosons on a lattice

- $\hat{a}_x$, $\hat{a}_x^\dagger$ annihilation and creation operators of a bosonic particle at site x. They satisfy canonical commutation relations $[\hat{a}_x, \hat{a}_y] = [\hat{a}_x^\dagger, \hat{a}_y^\dagger] = 0$, $[\hat{a}_x, \hat{a}_y^\dagger] = \delta_{x,y}$ for any x and y

- Φ_{vac} the state with no particles on the lattice. We have $\hat{a}_x \Phi_{\mathrm{vac}} = 0$ for any x.

- The Hilbert space with N bosons is spanned by the basis states $\hat{a}_{x_1}^\dagger \hat{a}_{x_2}^\dagger \cdots \hat{a}_{x_N}^\dagger \Phi_{\mathrm{vac}}$ with any $x_1, x_2, \ldots, x_N$.

Electrons on a lattice

- $\hat{c}_{x,\sigma}$, $\hat{c}_{x,\sigma}^\dagger$ annihilation and creation operators of an electron at site x with spin $\sigma \in \{\uparrow, \downarrow\}$. They satisfy canonical anticommutation relations $\{\hat{c}_{x,\sigma}, \hat{c}_{y,\tau}\} = \{\hat{c}_{x,\sigma}^\dagger, \hat{c}_{y,\tau}^\dagger\} = 0$, $\{\hat{c}_{x,\sigma}, \hat{c}_{y,\tau}^\dagger\} = \delta_{x,y}\delta_{\sigma,\tau}$ for any x, y, σ, and τ.

- Φ_{vac} the state with no particles on the lattice. We have $\hat{c}_{x,\sigma}\Phi_{\mathrm{vac}} = 0$ for any x, σ.

- The Hilbert space with N electrons is spanned by the basis states $\hat{c}_{x_1,\sigma_1}^\dagger \hat{c}_{x_2,\sigma_2}^\dagger \cdots \hat{c}_{x_N,\sigma_N}^\dagger \Phi_{\mathrm{vac}}$ with any $x_1, x_2, \ldots, x_N$ and $\sigma_1, \ldots, \sigma_N$.