時間依存 Lindblad 方程式における定常状態の分類と対称性

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H. Yoshida, Phys. Rev. A **109**, 022218 (2024)
H. Yoshida and R. Hamazaki, arXiv: 2602.13095

東京大学
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Outline

- **Introduction**
- **Main results**
- **Time-dependent GKSL equations**
  - Strong symmetry in the Schrödinger picture
  - Strong symmetry in the interaction picture
  - Criterion for the uniqueness of steady states
  - Time-dependent steady states in quasiperiodic systems
- **Summary and outlook**

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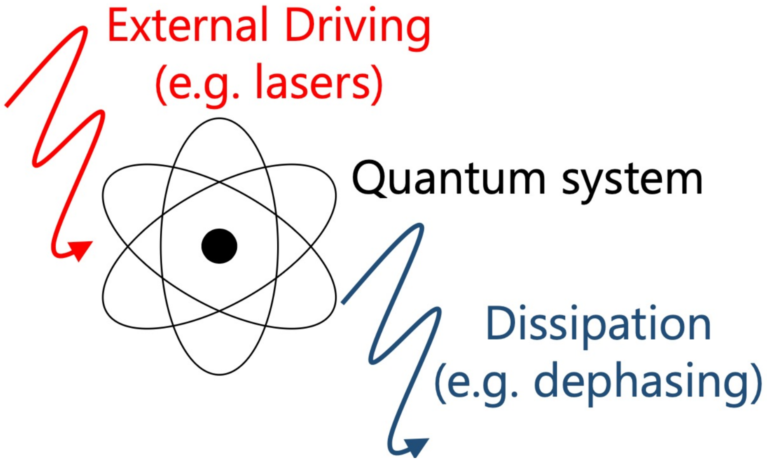
- **Summary and outlook**

Time dependent driven open quantum systems

- Realized with
  - Cold atoms
  - Trapped ions
  - Superconducting qubits
  - Diamond NV center……

M. Fitzpatrick, et al., PRX (2017).
J. Zhang et al., Nature **543**, 217 (2017).
S. Choi, et al., Nature **543**, 221 (2017).



- Possibility of exotic asymptotic states
  - Discrete dissipative time crystals
  - High-frequency expansions

Z. Gong, R. Hamazaki, M. Ueda, PRL (2018)
T. N. Ikeda and M. Sato, Sci. Adv. (2020)

Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation

$$\frac{d\rho_t}{dt} = \mathcal{L}_t[\rho_t] = -i[H_t, \rho_t] + \sum_m \left(L_{m,t}\rho_t L_{m,t}^\dagger - \frac{1}{2}\left\{ L_{m,t}^\dagger L_{m,t}, \rho_t \right\} \right)$$

ρ : density matrix H : Hamiltonian L : jump operators $\mathcal{L}$:Liouvillian

- GKSL equation describes Markovian dynamics of open quantum systems
  - ・time-independent case $H_t = H,\ L_{m,t} = L_m$
  - ・time-periodic case $H_t = H_{t+T},\ L_{m,t} = L_{m,t+T}$
  - ・time-quasiperiodic case (e.g. multi-frequency drive) $H_t \simeq H_{t+T},\ L_{m,t} \simeq L_{m,t+T}$

Steady states of GKSL equations

- **Steady states:** asymptotic states realized with **some** initial condition.
  In this talk, we define that the **steady states can depend on time,**
  i.e., a time-dependent state ρ_t^* is a steady state if

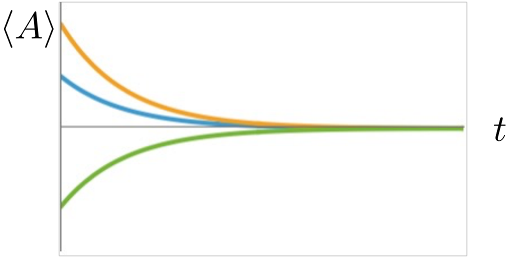
$$\lim_{t\to\infty}(\rho_t - \rho_t^*) = 0$$

  for some initial condition.

- **Time-dependent steady states are interesting!**
  - Synchronization
  - Dissipative time crystal

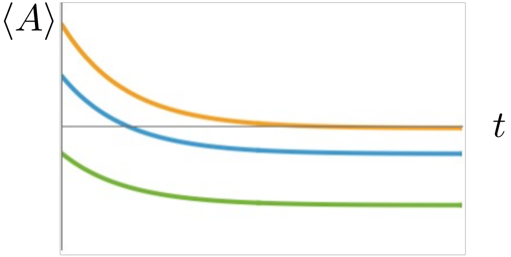
Q1: What are possible steady states for a given GKSL?

- **Unique steady states**
  - Every initial state relaxes to it
  - System *forgets* the information of the initial state

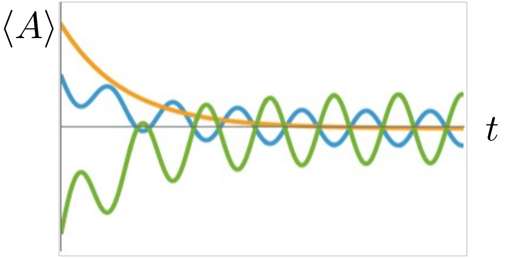


- **Multiple steady states**
  - Asymptotic state depends on initial state
  - System *remembers* some information

  Symmetry seems relevant!



- **Existence of time-dependent steady states?**
  - **Related to some symmetries?**

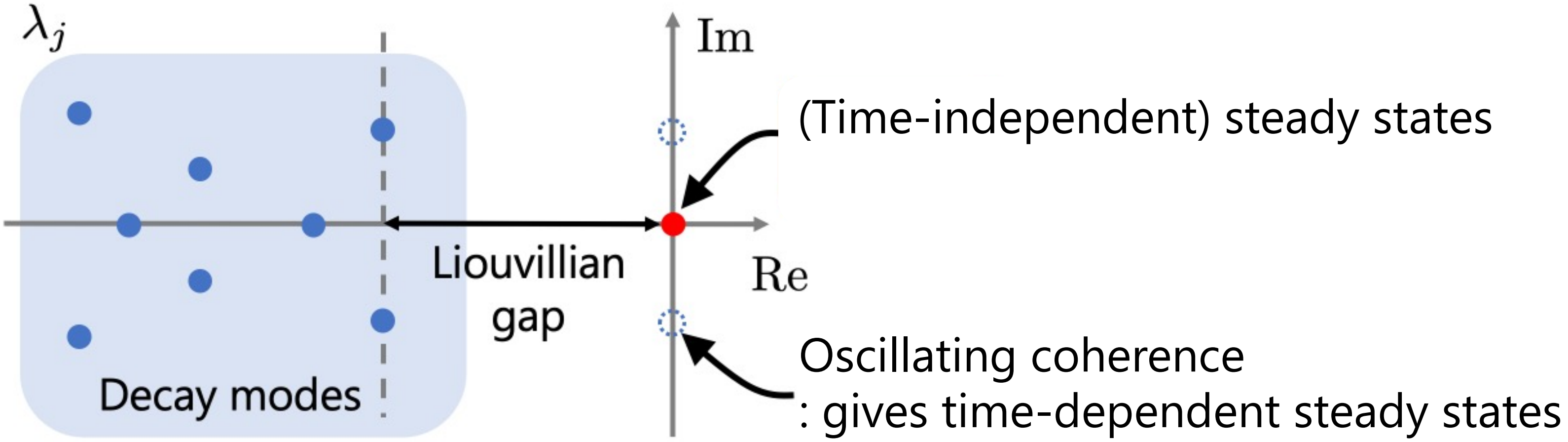


Time-independent GKSL equation

$$\partial_t \rho = \mathcal{L}(\rho) = -i[H, \rho] + \sum_m \left(L_m \rho L_m^\dagger - \frac{1}{2} \left\{ L_m^\dagger L_m, \rho \right\} \right)$$

■ Eigenvalues and eigenmodes of the Liouvillian

$$\mathcal{L}(\rho_k) = \lambda_k \rho_k \quad \Rightarrow \quad \rho(t) = \sum_k c_k e^{\lambda_k t} \rho_k$$



Symmetry is the key to understand the steady states

■ Symmetry of *time-independent* GKSL equations

$$\partial_t \rho_t = \mathcal{L}[\rho_t] = -i[H, \rho_t] + \sum_m \left(L_m \rho_t L_m^\dagger - \frac{1}{2} \{L_m^\dagger L_m, \rho_t\} \right)$$

- **Strong symmetry** (↔ weak symmetry: not relevant in this talk)

$$[H, O] = [L_m, O] = [L_m^\dagger, O] = 0$$

B. Buča and T. Prosen, New J. Phys. (2012).
V. V. Albert and L. Jiang, PRA (2014).

⟹ There are multiple steady states ρ^* s.t. $\mathcal{L}[\rho^*] = 0$.

- c.f.) when all L_m are Hermitian,

$$[H, O] = [L_m, O] = 0 \iff \mathcal{L}[O] = 0$$

⟹ $\rho^* \propto \mathbb{I}$ is always a steady state

Example 1: two-level system with strong symmetry

Model

$$H = \frac{\omega}{2}\sigma^z \qquad L = \sqrt{\gamma}\sigma^z \text{ : dephasing} \qquad \mathcal{L}[\rho] = -\frac{i\omega}{2}[\sigma^z, \rho] + \gamma\left(\sigma^z \rho \sigma^z - \rho\right)$$

Strong symmetry: $\mathbb{I}, \sigma^z$

Eigenmodes

$$\mathcal{L}[\sigma^\pm] = \mp i\omega - 2\gamma\sigma^\pm$$ The interference term disappears

$$\mathcal{L}[\mathbb{I}] = \mathcal{L}[\sigma^z] = 0$$ **Steady state: doubly degenerate**

- Linear combinations of steady states are also steady states
- So, the general form of steady states are

$$\begin{bmatrix} a & 0 \\ 0 & 1-a \end{bmatrix} \quad (0 \leq a \leq 1)$$: depend on the initial state

Example 2: two-level system without strong symmetry

Model

$$H = \frac{\omega}{2}\sigma^x \qquad L = \sqrt{\gamma}\sigma^z \text{ : dephasing} \qquad \mathcal{L}[\rho] = -\frac{i\omega}{2}[\sigma^x, \rho] + \gamma\left(\sigma^z \rho \sigma^z - \rho\right)$$

Strong symmetry: $\mathbb{I}$

Eigenmodes

$$\mathcal{L}[\sigma^x] = -2\gamma\sigma^x$$

$$\mathcal{L}[\sigma^z + \xi\sigma^y] = \omega\xi(\sigma^z + \xi\sigma^y) \qquad \xi = \frac{-\omega \pm \sqrt{\omega^2 - 8\gamma\omega}}{2\omega}$$

$\mathcal{L}[\mathbb{I}] = 0$ **Steady state: unique**

Q2: Efficient way to prove the uniqueness of steady states?

- Time-independent GKSL equations→ Long history of study

H. Spohn, Lett. Math. Phys. **2**, 33 (1977), Rev. Mod. Phys. **52**, 569 (1980).
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F. Fagnola and R. Rebolledo, J. Math. Phys. **43**, 1074 (2002).
B. Baumgartner, H. Narnhofer, and W. Thirring, J. Phys. A: Math. Theor. **41**, 065201 (2008).
D. Burgarth, G. Chiribella, V. Giovannetti, P. Perinotti, and K. Yuasa, New J. Phys. **15**, 073045 (2013).
D. Nigro, J. Stat. Mech. **2019**, 043202 (2019).
H. Yoshida, Phys. Rev. A **109**, 022218 (2024).
Y. Zhang and T. Barthel, J. Phys. A: Math. Theor. **57**, 115301 (2024)
D. Amato and P. Facchi, Sci Rep **14**, 14366 (2024)

- Time-dependent GKSL equations→ general results remain scarce

P. Menczel and K. Brandner, J. Phys. A: Math. Theor. **52**, 43LT01 (2019).
G. Di Meglio, D. Chruściński, K. Audenaert, M. B. Plenio, S. F. Huelga, arXiv:2410.14313

Criterion for uniqueness of steady states

- **For Hermitian jump operators** $\mathcal{H}$: Hilbert space $\dim \mathcal{H} = d$

The steady state is unique if and only if the set of operators $\{\mathbb{I}, H, L_1, \ldots, L_M\}$ generates all the operators on the Hilbert space under multiplication, addition, and scalar multiplication.

- **Example:**

$$H = \frac{\omega}{2}\sigma^x \quad L = \sqrt{\gamma}\sigma^z$$

The set of operators $\{\mathbb{I}, H, L\}$ generates all the operators!

$H \propto \sigma^x$ $HL \propto \sigma^y$ $L \propto \sigma^z$ → The steady state is unique.

Similar results for non-Hermitian jump operators

The steady state is unique and full-rank if and only if the set of operators $\left\{\mathbb{I}, H - \frac{i}{2}\sum_{m=1}^{M} L_m^\dagger L_m, L_1, \ldots, L_M\right\}$ generates all the operators under multiplication, addition, and scalar multiplication

F. Fagnola and R. Rebolledo, J. Math. Phys. **43**, 1074 (2002)
M. M. Wolf, Quantum Channels and Operations - Guided Tour(2012)
Hironobu Yoshida, Phys. Rev. A **109**, 022218 (2024)
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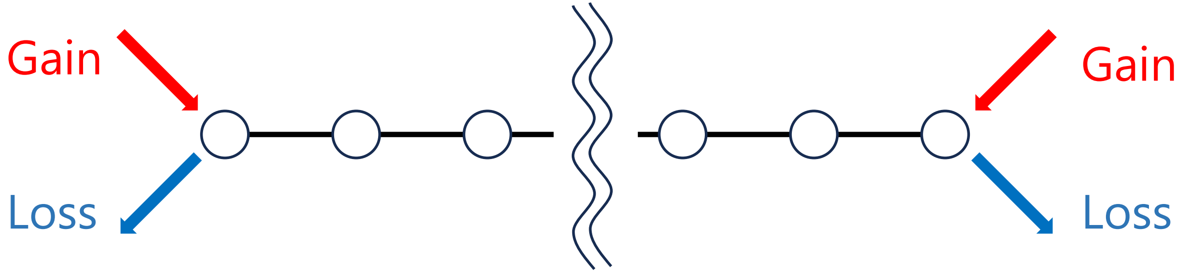
Example:



Motivations

- For *time-independent* GKSL equation, the relation between symmetry and steady states are extensively studied
- Generalization to *time-dependent* GKSL equations?
  - **Q1: What are possible steady states for a given GKSL?**
  - **Q2: Efficient way to prove the uniqueness of steady states?**
- We focus on time-dependent GKSL equation with Hermitian jump operators.

- $\mathcal{H}$: finite-dimensional Hilbert space with $\dim \mathcal{H} = d$

  $\mathcal{B}(\mathcal{H})$: the set of all the operators on $\mathcal{H}$

  $\mathcal{S}(\mathcal{H})$: the set of density operators on $\mathcal{H}$

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Main results (1/3)

- **For time-dependent GKSL equations, we define two distinct symmetries**

$$\mathcal{C}^{\mathrm{Sch}} = \{O \in \mathcal{B}(\mathcal{H}) \mid [H_t, O] = [L_{m,t}, O] = 0 \ \forall m, t\},$$

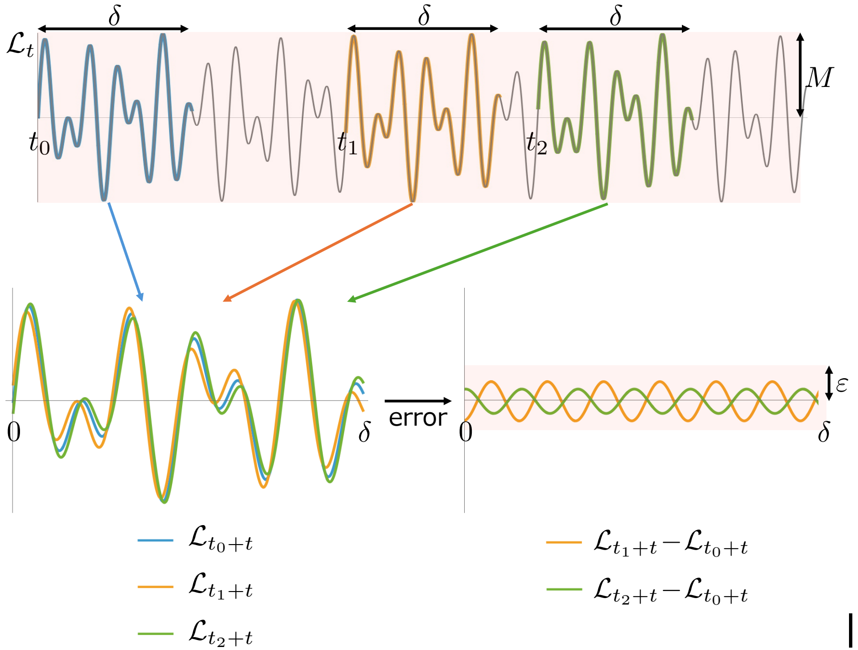
$$\mathcal{C}^{\mathrm{Int}} = \{O \in \mathcal{B}(\mathcal{H}) \mid [\tilde{L}_{m,t}, O] = 0 \ \forall m, t\}$$

$$U_t = \mathcal{T} e^{-i\int_0^t H_{t'} dt'}, \ \ \tilde{L}_{m,t} = U_t^\dagger L_{m,t} U_t.$$

$$\mathcal{C}^{\mathrm{Sch}} \subseteq \mathcal{C}^{\mathrm{Int}}$$

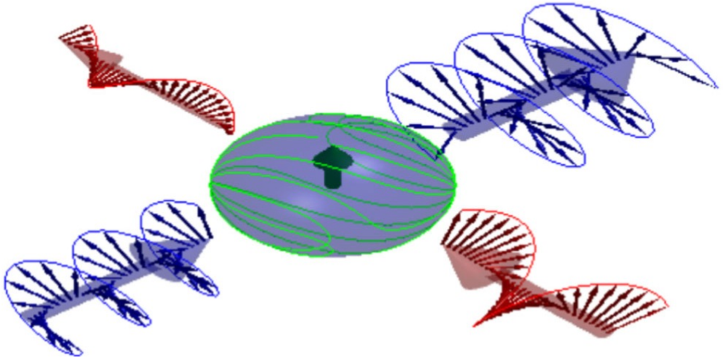
- **Using this symmetry, we classified the steady-state structure.**
  - Assumption: The Liouvillian is
    - continuous
    - bounded
    - quasi-periodic

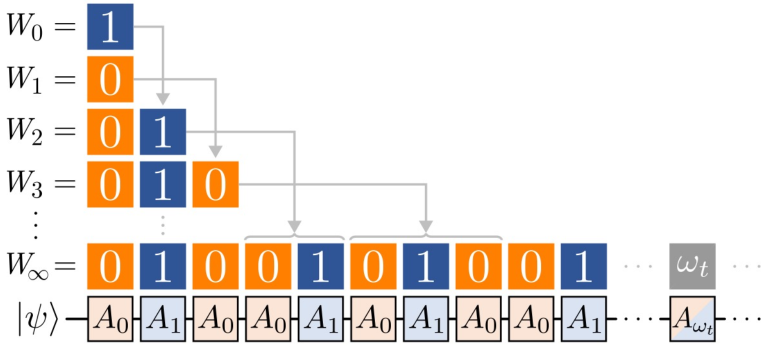
    Jump operators are Hermitian



Time-quasiperiodic driving

- Time-quasiperiodic driving includes:
  - Time-independent driving (no driving)
  - Time-periodic driving
  - Multi-frequency driving
  - Fibonacci drive

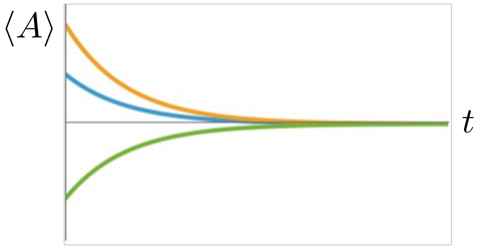
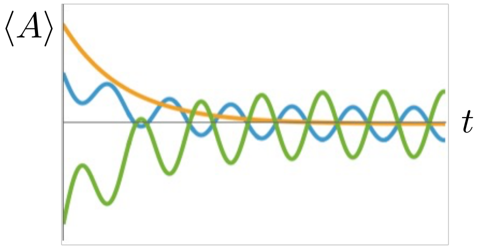
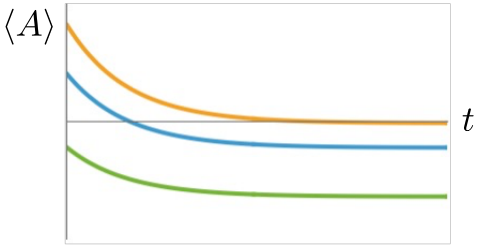
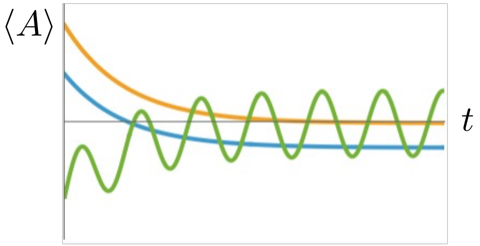




I. Martin, G. Refael, and B. Halperin PRX **7**, 041008 (2017)
S. P.-Cameo, C. B. Dag, W. W. Ho, and S. Choi, PRL **131**, 250401 (2023)

Main results (2/3)

Relation between two types of strong symmetries and steady states

		$\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}}$	
		$= \emptyset$	$\neq \emptyset$
$\mathcal{C}^{\mathrm{Sch}}$	$= \{c\mathbb{I}\}$	**(i) Unique steady state** $\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I}\}$ 	**(iv) Time-dependent steady states without non-trivial time-independent ones** 
	$\neq \{c\mathbb{I}\}$	**(ii) Multiple time-independent steady states without time-dependent ones** 	**(iii) Multiple time-independent and time-dependent steady states** 

Main results (3/3)

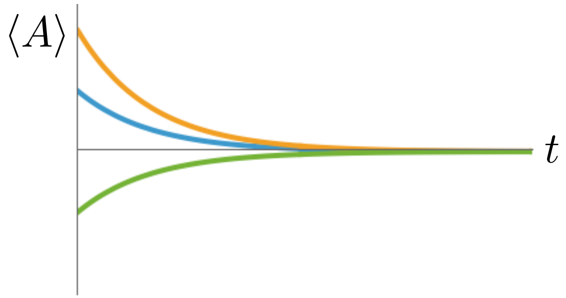
- **Practical criterion for the uniqueness of steady states**

$\mathcal{B}(\mathcal{H})$: set of operators on a Hilbert space $\mathcal{H}$

$$\mathrm{ad}_t[A] := i[H_t, A] + \partial_t A.$$

$\mathcal{A}_t^{\mathrm{ad}}$: subset of $\mathcal{B}(\mathcal{H})$ generated by sum, multiplication, and scaler multiplication of $\langle \mathbb{I}, \{\mathrm{ad}_t^n[L_{m,t}]\}_{m=1,\ldots,M;n=0,1,\ldots} \rangle$.

$$\exists t_0 \ \mathcal{A}_{t_0}^{\mathrm{ad}} = \mathcal{B}(\mathcal{H}) \iff$$



Unique steady state

Examples

- **Driven two-level system with dephasing**

$$H_t = \sigma^z + B\cos(\omega t)\sigma^x,\ L_t = \sqrt{\kappa}\sigma^z$$

$\Longrightarrow$
$$L_t|_{t=0} \propto \sigma^z$$
$$\mathrm{ad}_t(L_t)|_{t=0} = (i[H_t, L_t] + \partial_t L_t)|_{t=0} \propto \sigma^y$$

$\Longrightarrow$ $\mathbb{I}$, σ^z and σ^y generate all the 2×2 matrices

$(\sigma^x \propto \sigma^y \sigma^z)$

$\Longrightarrow$ $\lim_{t\to\infty} \rho_t = \mathbb{I}/2$ for any initial state ρ_0

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Strong symmetry in the Schrödinger picture

$$\mathcal{L}_t(\rho_t) = -i[H_t, \rho_t] + \sum_{m=1}^{M} \left(L_{m,t}\rho_t L_{m,t} - \frac{1}{2}\left\{ (L_{m,t})^2, \rho_t \right\} \right).$$

- For time-independent case, it holds that

$$[H, O] = [L_m, O] = 0 \quad \Longleftrightarrow \quad \mathcal{L}[O] = 0$$

- Straightforward generalization: strong symmetry in the Schrödinger picture

$$\mathcal{C}^{\mathrm{Sch}} = \{ O \in \mathcal{B}(\mathcal{H}) \mid [H_t, O] = [L_{m,t}, O] = 0 \ \forall m, t \},$$

$$\rho^* \in \mathcal{C}^{\mathrm{Sch}} \quad \Longleftrightarrow \quad \mathcal{L}_t[\rho^*] = 0 \ \forall t$$

- Thus, elements of $\mathcal{C}^{\mathrm{Sch}} \cap \mathcal{S}(\mathcal{H})$ give a time-independent steady states.
- But is the reverse true?

Strong symmetry in the Schrödinger picture

- Under the assumption that the Liouvillian is continuous, bounded, and quasi-periodic, Hermitian jump, the reverse also holds!

Theorem: Consider the time-dependent GKSL equation that satisfies the conditions above. Let $\rho^* \in \mathcal{S}(\mathcal{H})$ be a time-independent density matrix. Then, the following statements are equivalent:

1. $\lim_{t\to\infty} \rho_t = \rho^*$ for some initial condition ρ_0.
2. $\rho^* \in \mathcal{C}^{\mathrm{Sch}} \cap \mathcal{S}(\mathcal{H})$.

- But how about the time-dependent steady states?

Time dependent steady states?

- Even if $\mathcal{C}^{\rm Sch} = \{c\mathbb{I}\}$, there can be time-dependent steady states!

- **Counterexample:**

$$H_t = \frac{\omega}{2}\sigma^z,\ L_t = \sqrt{\kappa}[\sigma^x \cos(\omega t) + \sigma^y \sin(\omega t)],$$

$\mathcal{C}^{\rm Sch} = \{c\mathbb{I}\}$ $\Longrightarrow$ Time-independent steady state is unique

However, there exists a time-dependent steady states

$$\rho_t^* = \frac{1}{2}\begin{pmatrix} 1 & ae^{-i\omega t} \\ ae^{i\omega t} & 1 \end{pmatrix}$$

$(-1 \leq a \leq 1)$: parameter that depends on the initial condition

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Strong symmetry in the interaction picture

- We saw that $\mathcal{C}^{\mathrm{Sch}}$ does not characterize time-dependent steady states.
- What characterizes such steady states?
- To this end, we consider the GKSL equation *in the interaction picture*

$$U_t = \mathcal{T} e^{-i\int_0^t H_{t'}dt'},\ \tilde{\rho}_t = U_t^\dagger \rho_t U_t,\ \tilde{L}_{m,t} = U_t^\dagger L_{m,t} U_t$$

$$\frac{d\tilde{\rho}_t}{dt} = \tilde{\mathcal{L}}_t(\tilde{\rho}_t) = \sum_m \left(\tilde{L}_{m,t}\tilde{\rho}_t\tilde{L}_{m,t} - \frac{1}{2}\left\{ (\tilde{L}_{m,t})^2, \tilde{\rho}_t \right\} \right)$$

Strong symmetry in the interaction picture

- Previous counterexample

$$H_t = \frac{\omega}{2}\sigma^z,\ L_t = \sqrt{\kappa}[\sigma^x \cos(\omega t) + \sigma^y \sin(\omega t)],$$

$$\Longrightarrow \tilde{L}_t = \sqrt{\kappa}\sigma^x \quad \Longrightarrow \quad [\sigma^x, \tilde{L}_t] = 0 \text{ for all } t$$

- **Strong symmetry in the interaction picture**

$$\mathcal{C}^{\mathrm{Int}} = \{O \in \mathcal{B}(\mathcal{H}) \mid [\tilde{L}_{m,t}, O] = 0\ \forall m, t\}$$

- $\tilde{\rho}^* \in \mathcal{C}^{\mathrm{Int}} \cap \mathcal{S}(\mathcal{H}) \Longrightarrow \rho_t^* = U_t \tilde{\rho}^* U_t^\dagger$ is a (generally time-dependent) steady state
- $\mathcal{C}^{\mathrm{Int}}$ is weaker than $\mathcal{C}^{\mathrm{Sch}}$: $\mathcal{C}^{\mathrm{Sch}} \subseteq \mathcal{C}^{\mathrm{Int}}$

Strong symmetry in the interaction picture

- Surprisingly, the existence of **non-trivial steady state** is equivalent to the existence of **non-trivial strong symmetry in the interaction picture**
- Assumption: continuous, bounded, and quasi-periodic, Hermitian jump

Theorem: Consider the time-dependent GKSL equation that satisfies the conditions above. Then, the following statements are equivalent:

1. $\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I} \mid c \in \mathbb{C}\}$.
2. For any initial state ρ_0, the solution ρ_t of the GKSL equation converges to the maximally mixed state:

$$\lim_{t\to\infty} \rho_t = \mathbb{I}/d.$$

as the unique steady state.

Existence of time-dependent steady states

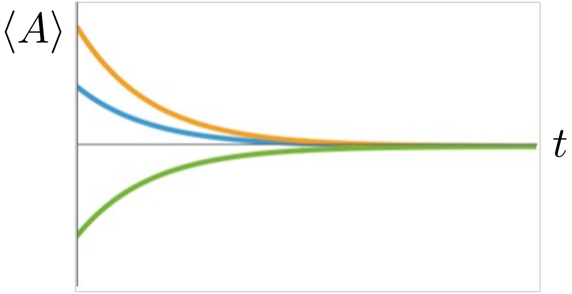
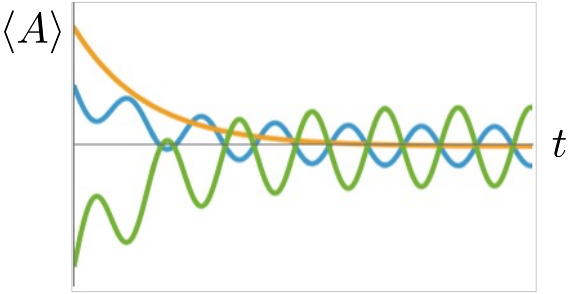
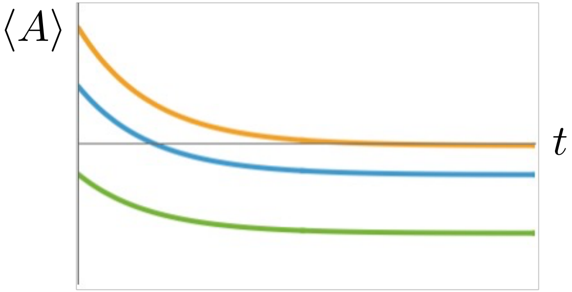
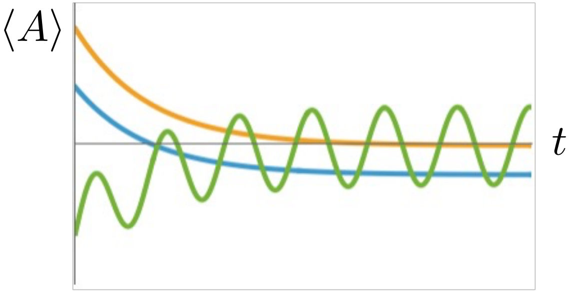
- What if $\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}} \neq \emptyset$?
- There are time dependent steady states if and only if $\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}} \neq \emptyset$!

Theorem: Consider the time-dependent GKSL equation that satisfies the conditions above. Then, the following statements are equivalent:

1. $\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}} \neq \emptyset$.
2. There exists a time-dependent steady state, i.e., there is a time-dependent ρ_t^* that does not converge to any time-independent density matrix as $t \to \infty$ such that

$$\lim_{t\to\infty} \rho_t = \rho_t^*,$$

   for some initial condition ρ_0.

		$\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}}$	
		$= \emptyset$	$\neq \emptyset$
$\mathcal{C}^{\mathrm{Sch}}$	$= \{c\mathbb{I}\}$	**(i) Unique steady state** $\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I}\}$ 	**(iv) Time-dependent steady states without non-trivial time-independent ones** 
	$\neq \{c\mathbb{I}\}$	**(ii) Multiple time-independent steady states without time-dependent ones** 	**(iii) Multiple time-independent and time-dependent steady states** 

Example: dynamical strong symmetry

- Even for *time-independent* GKSL equations, $\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}} \neq \emptyset$ in general!

- Strong dynamical symmetry B. Buča, J. Tindall, and D. Jaksch, Nat Commun **10**, 1730 (2019).

$$[H, A] = \Omega A \text{ and } [L_m, A] = 0 \ \forall m \qquad \Omega \in \mathbb{R}$$

$$\Longrightarrow \mathcal{L}[A] = -i\Omega \qquad \mathcal{L}[A^\dagger] = i\Omega$$

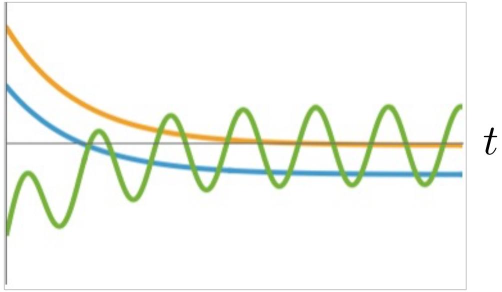
Oscillating coherence →time-dependent steady states

- $A \notin \mathcal{C}^{\mathrm{Sch}}$ but $A \in \mathcal{C}^{\mathrm{Int}}$!

$$A^\dagger A \in \mathcal{C}^{\mathrm{Sch}}$$

$\Longrightarrow$

(iii) Multiple time-independent and time-dependent steady states



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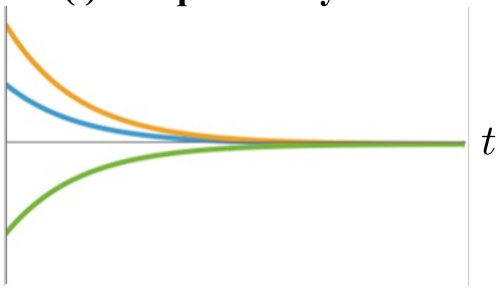
Criterion for the uniqueness of steady states

- We have proven that

$$\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I} \mid c \in \mathbb{C}\} \quad \Longleftrightarrow$$

(i) Unique steady state



- But how do we calculate $\mathcal{C}^{\mathrm{Int}}$?

$$\mathcal{C}^{\mathrm{Int}} = \{O \in \mathcal{B}(\mathcal{H}) \mid [\tilde{L}_{m,t}, O] = 0 \ \forall m, t\}$$

$$U_t = \mathcal{T} e^{-i\int_0^t H_{t'}\,dt'}, \ \ \tilde{L}_{m,t} = U_t^\dagger L_{m,t} U_t.$$

- Idea: instead of directly calculating $\tilde{L}_{m,t} = U_t^\dagger L_{m,t} U_t$, we consider the time derivative of it.

Criterion for the uniqueness of steady states

- We focus on the time derivative of $\{\tilde{L}_{m,t}\}$

$$\partial_t^n \tilde{L}_{m,t} = U_t^\dagger \, \mathrm{ad}_t^n (L_{m,t}) U_t$$

$\mathrm{ad}_t(A) := i[H_t, A] + \partial_t A$: time-dependent adjoint operation

$$O \in \mathcal{C}^{\mathrm{Int}} \quad \Longrightarrow \quad [O, \partial_t^n \tilde{L}_{m,t}] \ \forall m, n$$

- If we fix $t = t_0$, U_{t_0} is just a common unitary transformation

- We consider

$$\mathcal{C}_t^{\mathrm{ad}} = \{O \in \mathcal{B}(\mathcal{H}) \mid [\mathrm{ad}_t^n(L_{m,t}), O] = 0 \ \forall m, n\}$$

$\mathcal{C}_{t=t_0}^{\mathrm{ad}} = \{c\mathbb{I} \mid c \in \mathbb{C}\}$ for some t_0 → $\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I} \mid c \in \mathbb{C}\}$

- Note that we do not have to calculate all n. Typically n=0,1 are necessary.

Criterion for the uniqueness of steady states

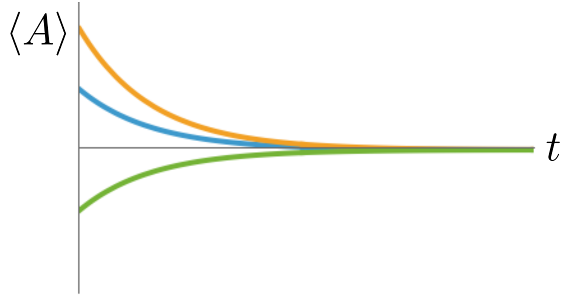
- Instead of $\mathcal{C}_t^{\mathrm{ad}}$, it is easier to calculate $\mathcal{A}_t^{\mathrm{ad}}$ in general.

  $\mathcal{A}_t^{\mathrm{ad}}$: subset of $\mathcal{B}(\mathcal{H})$ generated by sum, multiplication, and scaler multiplication of $\langle \mathbb{I}, \{\mathrm{ad}_t^n[L_{m,t}]\}_{m=1,\ldots,M;n=0,1,\ldots} \rangle$.

- Again, we do not have to calculate all n. Typically n=0,1 are necessary.

- If H_t and $L_{m,t}$ are analytical at all t, the following equivalence relation holds.

$$\exists t_0 \ \mathcal{A}_{t_0}^{\mathrm{ad}} = \mathcal{B}(\mathcal{H}) \iff$$



Unique steady state

Outline

- Introduction
- Main results
- Time-dependent GKSL equations
  - Strong symmetry in the Schrödinger picture
  - Strong symmetry in the interaction picture
  - Criterion for the uniqueness of steady states
  - Time-dependent steady states in quasiperiodic systems
- Summary and outlook

Time-dependent steady states in the quasi-periodic case

- As we have seen, there is time-dependent steady states if and only if $\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}} \neq \emptyset$

- Many-body example that can be realized experimentally?
  →Hubbard model with multi-frequency drives

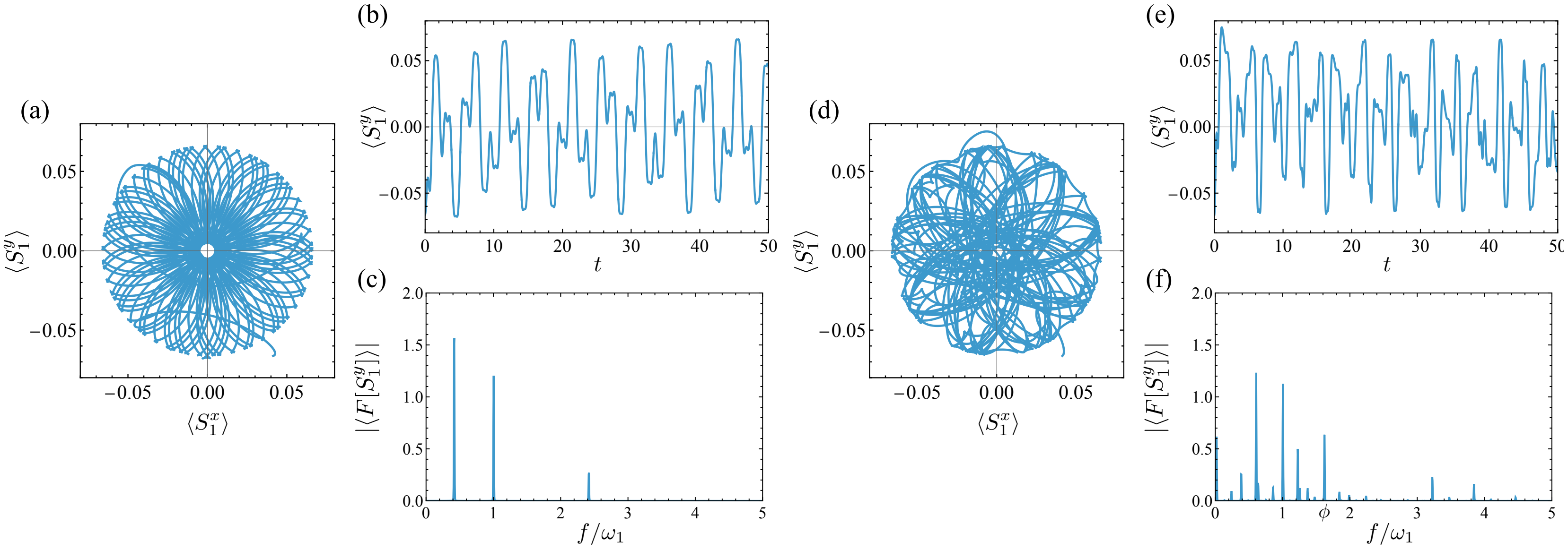
$$H_t = -\tau \sum_{\langle j,j' \rangle} \sum_{\sigma=\uparrow,\downarrow} \left(c^\dagger_{j,\sigma} c_{j',\sigma} + \mathrm{h.c.} \right) + \sum_j \left[U n_{j,\uparrow} n_{j,\downarrow} + \mu_j n_j \right]$$
$$+ \frac{B_1}{2} \left[e^{-i\omega_1 t} S^+ + e^{i\omega_1 t} S^- \right] + \frac{B_2}{2} \left[e^{-i\omega_2 t} S^+ + e^{i\omega_2 t} S^- \right],$$
$$L_j = \sqrt{\kappa_j} n_j .$$

⟹ $S^+, S^- \in \mathcal{C}^{\mathrm{Int}}$ but $S^+, S^- \notin \mathcal{C}^{\mathrm{Sch}}$

Single frequency case: K. Chinzei and T. N. Ikeda, PRL **125**, 060601 (2020)

Numerical result

- (a-c) single-frequency drive
- (d-f) two-frequency drive



- Time-dependent steady states in quasiperiodic systems!

Outline

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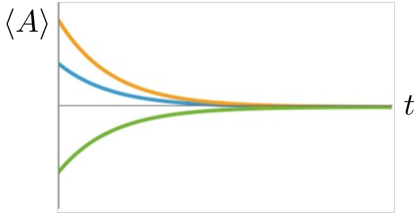
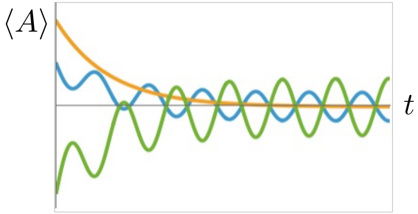
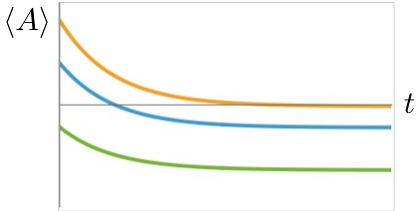
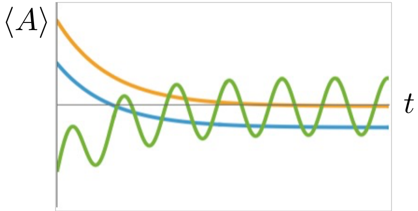
- **Summary and outlook**

Summary

- We classified the asymptotic behavior of time-quasiperiodic GKSL under the assumption of Hermitian jump operators by introducing two symmetries

<table>
<tr><td></td><td></td><td colspan="2">$\mathcal{C}^{\mathrm{Int}} \setminus \mathcal{C}^{\mathrm{Sch}}$</td></tr>
<tr><td></td><td></td><td>$= \emptyset$</td><td>$\neq \emptyset$</td></tr>
<tr><td rowspan="2">$\mathcal{C}^{\mathrm{Sch}}$</td><td>$= \{c\mathbb{I}\}$</td><td>(i) Unique steady state
$\mathcal{C}^{\mathrm{Int}} = \{c\mathbb{I}\}$
</td><td>(iv) Time-dependent steady states without non-trivial time-independent ones
</td></tr>
<tr><td>$\neq \{c\mathbb{I}\}$</td><td>(ii) Multiple time-independent steady states without time-dependent ones
</td><td>(iii) Multiple time-independent and time-dependent steady states
</td></tr>
</table>

Outlook

- Generalizations to non-Hermitian jump operators?
- Extension to non-quasiperiodic Liouvillians?
- Weak symmetry in the Schrödinger and interaction pictures?
- SSB of strong/weak symmetry in these pictures?

F. Minganti, A. Biella, N. Bartolo, and C. Ciuti, PRA **98**, 042118 (2018).
S. Lieu, R. Belyansky, J. T. Young, et.al., PRL **125**, 240405 (2020).
D. Gu, Z. Wang, and Z. Wang, arXiv:2406.19381
N. Ziereis, S. Moudgalya, and M. Knap, arXiv:2509.09669.