

# 部分可解性がもたらす非熱平衡化現象 — 孤立量子系から開放量子系まで —



[Phys. Rev. B 109, 104307 (2024), Phys. Rev. B 110, 224306 (2024),  
Phys. Rev. Research 7, 023099 (2025), arXiv:2512.11216]

統計物理学懇談会

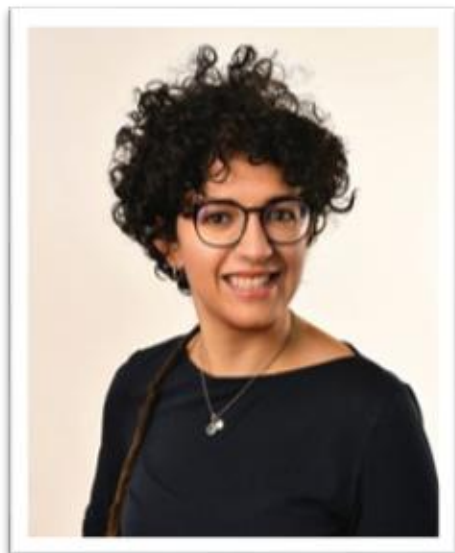
@オンライン 2026年3月12-13日

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# Collaborators



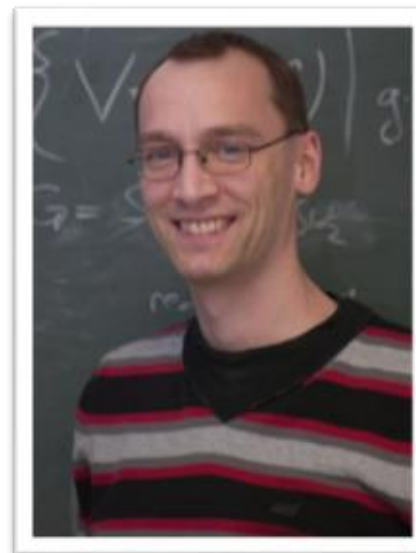
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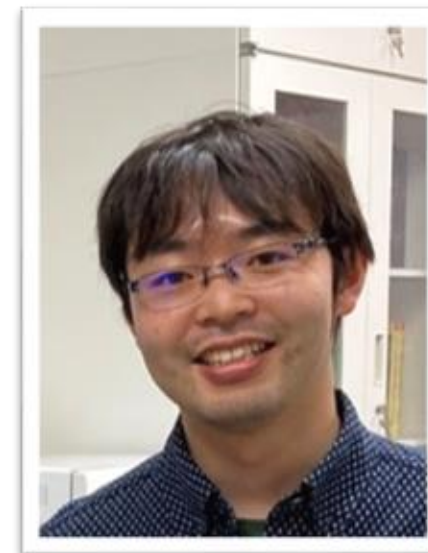
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# Outline



- Thermalization & Quantum many-body scars (QMBS)
  - Partially solvable isolated quantum systems
    - QMBS induced by restricted spectrum-generating algebra (rSGA)
    - QMBS induced by Hilbert space fragmentation (HSF)
  - Partially solvable open quantum systems
  - Concluding remarks
- Two main players inducing partial solvability!

# Outline

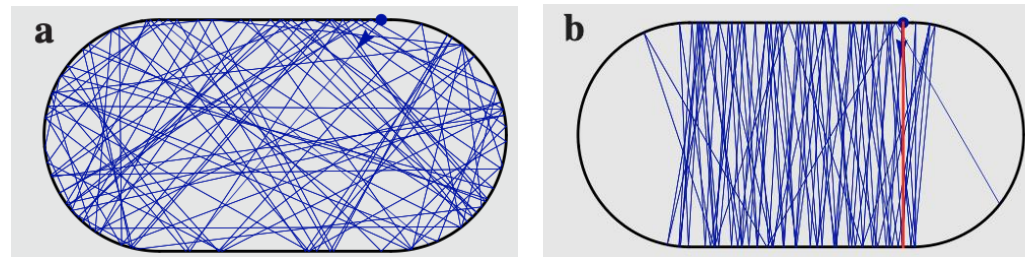


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# Thermalization & classical scars

- Classical thermalization is understood based on
  - Principle of equal probability  
(All microstates in the energy shell shows up with equal probability)
  - Typicality  
(Almost all microstates are macroscopically indistinguishable from thermal states. )
- Exceptional non-thermal microstate (**scar**) shows up under the special initial condition.

e.g. Scar in stadium billiard  
[Serbyn et al. (2021)]



- a. starting away from unstable periodic trajectory
- b. starting from near unstable periodic trajectory

# Thermalization & QMBS



- Quantum thermalization is understood based on

- Ergodicity in Hilbert space

$$\overline{|\langle O_{\text{macro}} \rangle_t - \langle O_{\text{macro}} \rangle_t|^2} \leq \frac{\|O_{\text{macro}}\|^2}{D_{\text{eff}}}, \quad D_{\text{eff}} := \left( \sum_j |c_j|^4 \right)^{-1}$$

⇒ **The initial state needs to be prepared as the superposition of many enough energy eigenstates.**

[Riemann (2008)]

$$\overline{x(t)} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt x(t)$$

$$\langle O \rangle_t := \langle \psi(t) | O | \psi(t) \rangle, \quad |\psi(0)\rangle = \sum_{E_j \in (E^* - \delta E, E^*]} c_j |E_j\rangle$$

- Eigenstate thermalization hypothesis (ETH)

[Deutsch (1991), Srednicki (1994)]

[Iyoda et al. (2017)]

$$\langle E_a | O_{\text{macro}} | E_a \rangle \underset{\text{TL}}{=} \langle O_{\text{macro}} \rangle_{\text{MC}}, \quad \text{for almost all } E_a \in (E^* - \delta E, E^*]$$

# Thermalization & QMBS

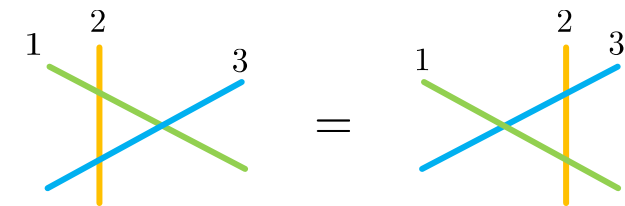


- Non-thermal energy eigenstates usually show up in a system which never thermalizes.

- Many-body localized systems [Nandkishore et al. (2015)]

- Integrable systems [Rigol et al. (2008)]

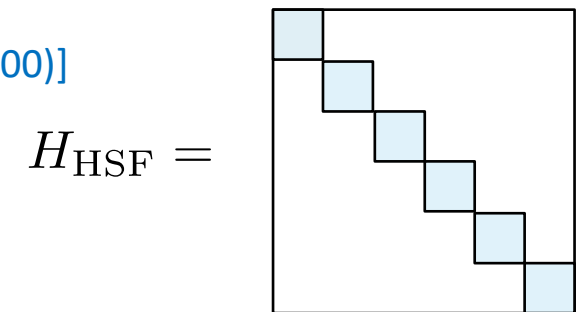
Infinitely many conserved quantities, relatively small EE



- Hilbert space fragmentation (HSF) [Nicolai (1976), Batista et al. (1995), Zhang et al. (1997), Batista-Ortiz (2000)]

Hidden non-local conserved quantities  
(Very often show up as kinetic constraints)

Hilbert space fragmented into exponentially many invariant subspaces



$$H_{\text{HSF}} =$$

**Violation of a certain kind of “uniformness” brings non-thermalization.**

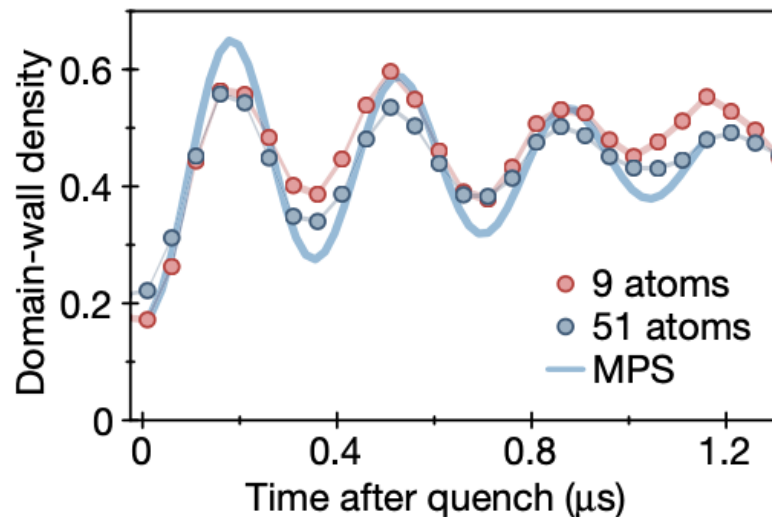
# Thermalization & QMBS



- Non-thermal energy sometimes show up in a thermalizing system.

“Quantum Many-Body Scars (QMBS)” [Shiraishi et al. (2017), Turner et al. (2017)] for theoretical aspects  
[Bernien et al. (2017)] for experiments

- Persistent oscillations



[Bernien et al. (2017)]

Domain-wall density after the quench from  $|\mathbb{Z}_2\rangle = |\bullet \circ \bullet \circ \dots\rangle$  on the Rydberg atom chain.

(● : Excited state; ○ : Ground state)

Long-lived oscillations

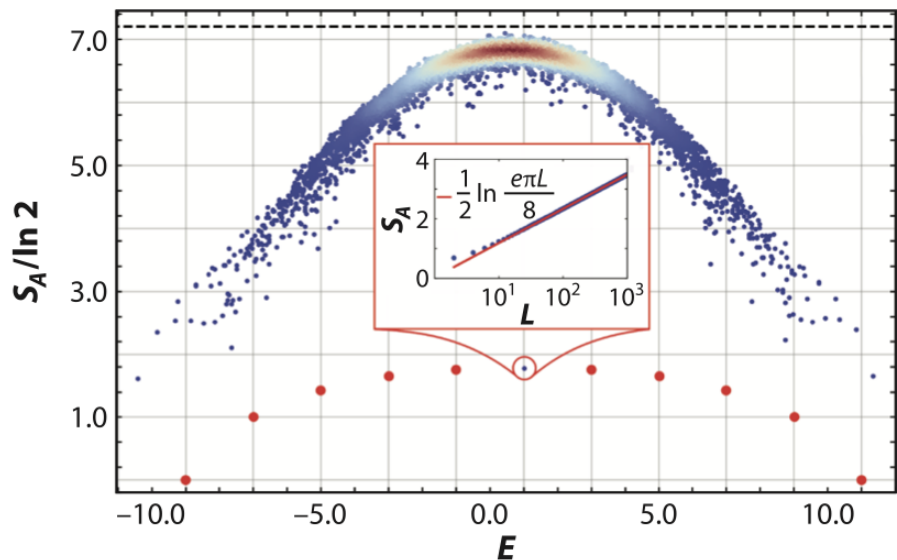
⇒ **Never thermalize! (No relaxation.)**

# Thermalization & QMBS

- Non-thermal energy sometimes show up in a thermalizing system.

“Quantum Many-Body Scars (QMBS)” [Shiraishi et al. (2017), Turner et al. (2017)] for theoretical aspects  
 [Bernien et al. (2017)] for experiments

- Small entanglement entropy (EE)



[Chandran et al. (2003)]

Half-chain entanglement entropy of the spin-1 XY model in zero-magnetization sector.

Relatively small entanglement entropy  $\sim o(V)$  compared to those of thermal pure states  $\sim O(V)$ .

⇒ **MPS with a finite bond dimension is a good benchmark for finding QMBS.**

$$S_{EE} := -\text{tr}(\rho' \log \rho') \leq \log \chi$$

$\chi$   
bond dimension of MPS

# Scar Hamiltonians



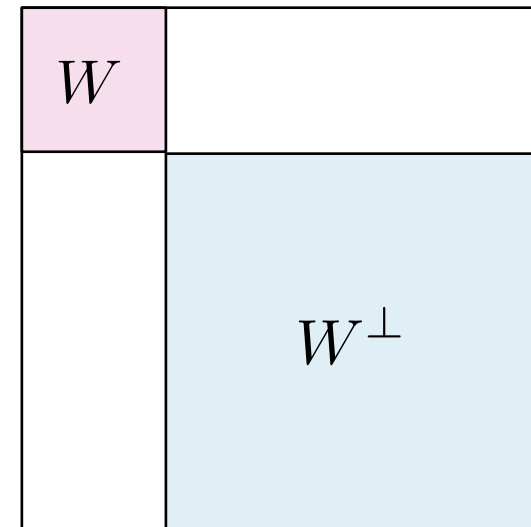
- Hamiltonian with QMBS

Partial solvability is closely related to the quantum scars.

- Hamiltonians with the block diagonal structure.

$$\mathcal{H} \simeq W \oplus W^\perp$$

↑ Thermalizing subspace  
↓ Relatively small (solvable) invariant subspace (Dynamically isolated)



- W protected by

Projector embeddings  $\Rightarrow$  Solvability is not necessary.  
[Shiraishi et al. (2017)]

Extra algebraic relation (e.g. rSGA)  $\Rightarrow$  Solvable if the reference state is solvable.  
[Vefek et al. (2017), Moudgalya et al. (2018), Moudgalya et al. (2020)]

Hilbert space fragmentation (HSF)  $\Rightarrow$  Solvability can be embedded (not necessary).  
[Pai et al. (2019), Sala et al. (2020), Khemani et al. (2020)]

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# W protected by rSGA

- Restricted spectrum-generating algebra (rSGA) [Arno et al. (1988), Yang (1989), Moudgalya et al. (2018)]

$$[H, Q] - \mathcal{E}Q|_W = 0, \quad W \subset \mathcal{H}, \quad Q : \text{Local operator}$$

- Extra algebraic relation that holds in W (not in the entire Hilbert space).
- If one finds a solvable state  $|\psi_0\rangle \in W$ , a tower of solvable states are constructed.

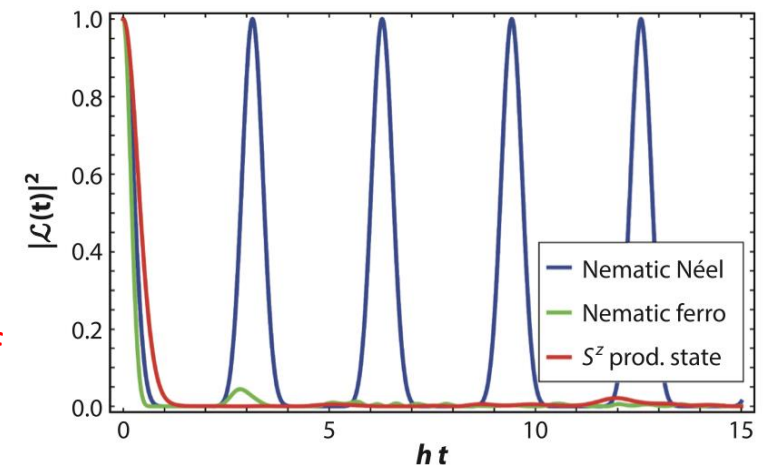
$$H|\psi_0\rangle = \mathcal{E}_0|\psi_0\rangle \Rightarrow HQ^n|\psi_0\rangle = (\mathcal{E}_0 + n\mathcal{E})Q^n|\psi_0\rangle$$



Equally-spacing energy spectrum in W.

$W = \text{span} \{Q^n|\psi_0\rangle\}_n$  is a solvable invariant subspace of  $H$ .

**Persistent oscillation** observed in dynamics of Loschmidt echo for the spin-1 XY.



[Chandran et al. (2023)]

# W protected by rSGA

- Example: spin-1 XY model with uniform magnetic fields [Schecter et al. (2019)]

$$H = \sum_{j=1}^L \mathbf{1} \otimes \cdots \otimes_{j,j+1} h \otimes \cdots \otimes \mathbf{1} \in \text{End}((\mathbb{C}^3)^L), \quad \mathbb{C}^3 = \text{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$h_{j,j+1} = \frac{J}{2}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + \frac{m}{2}(S_j^z + S_{j+1}^z), \quad m > 0$$

- Non-integrable spin chain (not all energy eigenstates are solvable).

- Trivial energy eigenstate (the ground state)

$$H|\Omega\rangle = -mL|\Omega\rangle, \quad |\Omega\rangle = |22\dots 2\rangle$$

- Exact quasiparticle excitations

$$H(Q^+)^n|\Omega\rangle = -m(L - 2n)(Q^+)^n|\Omega\rangle, \quad Q^\pm = \sum_{j=1}^L (-1)^j (S_j^\pm)^2$$

equally-spacing spectrum in W

local excitation (quasi particle)

Energy spectrum in  $W = \text{span}\{(Q^+)^n|\Omega\rangle\}_n$

$$\begin{aligned} & \vdots \\ & \underline{H(Q^+)^3|\Omega\rangle = -m(L - 6)(Q^+)^3|\Omega\rangle} \\ & \underline{H(Q^+)^2|\Omega\rangle = -m(L - 4)(Q^+)^2|\Omega\rangle} \\ & \underline{HQ^+|\Omega\rangle = -m(L - 2)Q^+|\Omega\rangle} \\ & \underline{H|\Omega\rangle = -mL|\Omega\rangle} \end{aligned}$$

# W protected by rSGA

- Example: AKLT-type model [Moudgalya et al. (2018), CM (2023)]

$$H = \sum_{j=1}^L \mathbf{1} \otimes \cdots \otimes_{j,j+1} h_{j,j+1} \otimes \cdots \otimes \mathbf{1} \in \text{End}((\mathbb{C}^3)^L), \quad \mathbb{C}^3 = \text{span}\{|0\rangle, |1\rangle, |2\rangle\}$$

$$h_{j,j+1} = P_{j,j+1}^{(2)} = \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2 \quad \text{: non-integrable spin chain}$$

- MPS ground state [Affleck et al. (1987)]

$$H|\psi_0\rangle = 0, \quad |\psi_0\rangle = \text{tr}_a(\vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A}) \\ = \sum_{\{m_1, \dots, m_N\} \in \{0,1,2\}^N} \text{tr}_a(A_{m_1} A_{m_2} \cdots A_{m_N}) |m_1, m_2, \dots, m_N\rangle$$

$$A_0 = \sqrt{\frac{2}{3}}\sigma_a^+, \quad A_1 = -\sqrt{\frac{1}{3}}\sigma_a^z, \quad A_2 = -\sqrt{\frac{2}{3}}\sigma_a^- \quad \text{: MPS with the 2-dim bond space}$$

- Exact quasiparticle excitations [Arovas (1989), Moudgalya et al. (2018)]

$$H(Q^\pm)^n |\psi_0\rangle = 2n(Q^\pm)^n |\psi_0\rangle, \quad Q^\pm = \sum_{j=1}^L (-1)^j (S_j^\pm)^2$$

equally-spacing spectrum in W

local excitation (quasi particle)

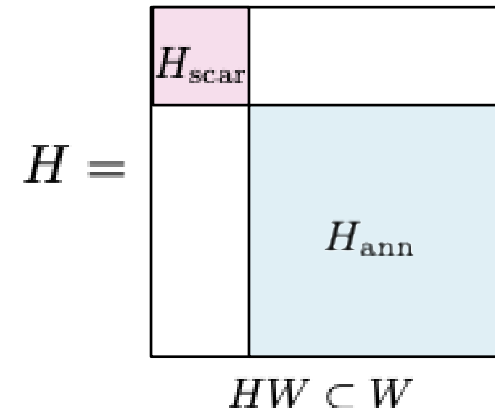
Does rSGA emerges in W?

# W protected by rSGA

- Hamiltonian with rSGA-induced scars [Serbyn et al. (2021)]

$$H = H_{\text{ann}} + H_{\text{scar}}, \quad H|_W = H_{\text{scar}}$$

$H_{\text{ann}}|_W = 0$  : annihilated in the scar subspace  $W = \text{span}\{(Q^+)^n|\Omega\rangle\}_n$



- Spin-1 XY model [Chandran et al. (2023)]

$$H_{\text{ann}} = \frac{J}{2} \sum_{j=1}^L (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y), \quad H_{\text{scar}} = m \sum_{j=1}^L S_j^z$$

$W = \text{span}\{(Q^+)^n|\Omega\rangle\}_n = \text{span}\{(Q^-)^n|\Omega\rangle\}_n$  : scar subspace ( $Q^\pm = \sum_{j=1}^L (-1)^j (S_j^\pm)^2$ )

$\Rightarrow [(S^+)^2, (S^-)^2] = 2S^z, \quad [S^z, (S^\pm)^2] = \pm(S^\pm)^2$  : hidden  $\text{su}(2)$  in  $W$   
 $(H_{\text{scar}}W \subset W, \quad Q^\pm W \subset W)$

$\Rightarrow [H, Q^\pm] \mp 2mQ^\pm|_W = 0$  : **rSGA emerges in W!**

# W protected by rSGA

- Hamiltonian with rSGA-induced scars [Serbyn et al. (2021)]

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$H_{\text{ann}}|_W = 0$  : annihilated in the scar subspace  $W = \text{span}\{(Q^+)^n|\Omega\rangle\}_n$

$$H = \begin{array}{|c|c|} \hline H_{\text{scar}} & \\ \hline & H_{\text{ann}} \\ \hline \end{array}$$

$HW \subset W$

- AKLT model [Moudgalya et al. (2018)]

$$H_{\text{AKLT}} = H_{\text{ann}} + \mathcal{E}H_{\text{scar}},$$

$h_{j,j+1}\vec{A} \otimes_p \vec{A} = 0$  : **frustration-free condition** (The lowest energy state of the local Hamiltonian is the gs of the entire Hamiltonian.)  
 $\Rightarrow |\psi_0\rangle = \text{tr}_a(\vec{A} \otimes_p \vec{A} \otimes_p \cdots \otimes_p \vec{A})$

$(h_{j,j+1} - \mathcal{E})(\vec{B} \otimes_p \vec{A} - \vec{A} \otimes_p \vec{B}) = 0$  : **local eigenstate condition**

$\vec{B} \otimes_p \vec{B} = 0, (S^+)^2\vec{B} = (S^+)^4\vec{A} = 0$  : **No double/adjacent occupation in W.  $\Rightarrow$  rSGA emerges in W**

$\Rightarrow |\psi_n\rangle = (Q^+)^n|\psi_0\rangle = \sum_{j_1, \dots, j_n} (-1)^{\sum_{k=1}^n j_k} \text{tr}_a(\vec{A} \otimes_p \cdots \otimes_p \vec{B}_{j_1} \otimes_p \cdots \otimes_p \vec{B}_{j_n} \otimes_p \cdots \otimes_p \vec{A}) \quad [H, Q^+] - 2\mathcal{E}Q^+|_W = 0$

# W protected by rSGA

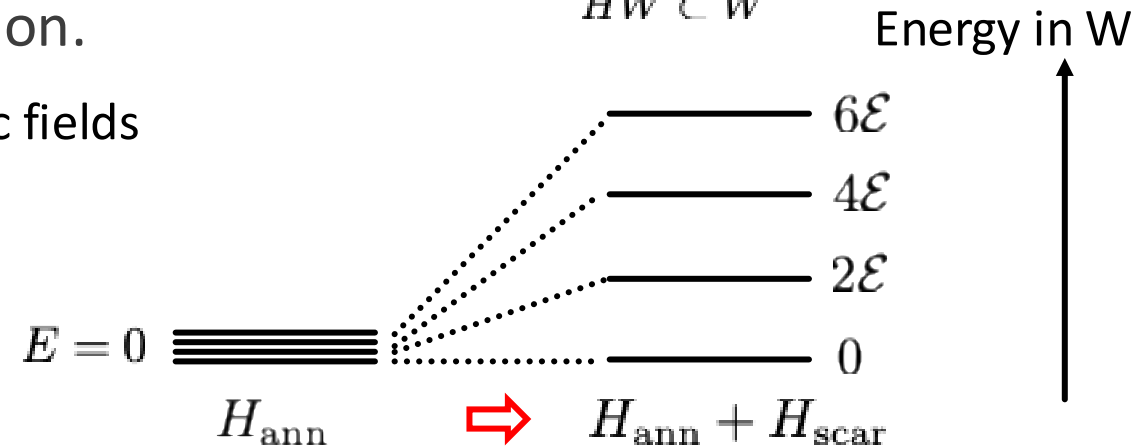
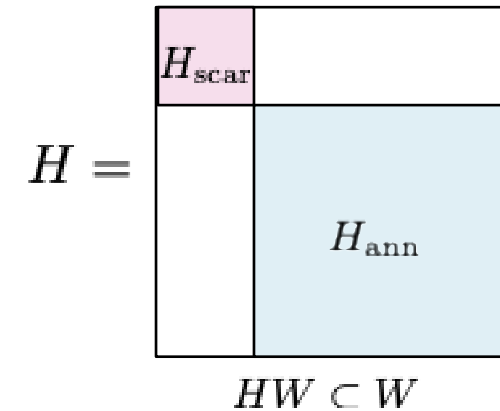
- Hamiltonian with rSGA-induced scars

$$H = H_{\text{ann}} + H_{\text{scar}}, \quad H|_W = H_{\text{scar}}$$

$H_{\text{ann}}|_W = 0$  : annihilated in the scar subspace  $W = \text{span}\{(Q^+)^n|\Omega\}_n$

- $H_{\text{ann}}$  has  $W$  as its big kernel (many degenerate gs).
- $H_{\text{scar}}$  splits the degenerate gs by rSGA relation.

Ex.) Cartan of  $\mathfrak{su}(2) \Rightarrow$  spin-1 XY with uniform magnetic fields  
 rSGA (single tower)  $\Rightarrow$  AKLT-type models  
 Onsager-alg. induced scars [Shibata et al. (2020)]



# Beyond rSGA



- Quantum scars with interacting quasiparticles [Chen (2023)]

- rSGA-induced quantum scars

$$|\psi_n\rangle = (Q^+)^n |\psi_0\rangle, \quad Q^+ = \sum_{j=1}^L (-1)^j (S_j^+)^2 \quad : \text{Identical \& non-interacting quasiparticles carrying momentum } \pi$$



$$Q^+(k) = \sum_{j=1}^L e^{ikj} (S_j^+)^2 \quad : \text{Quasiparticles carrying momentum } k \Rightarrow \text{interacting quasiparticles}$$

- Solvable quasiparticle excitations with interactions

$$|\psi_n\rangle = \sum_{1 \leq x_1 < \dots < x_n \leq L} \sum_{P \in \mathfrak{S}_n} A_n(P) e^{i \sum_{j=1}^n k_{P(j)} x_j} \text{tr}_a(\vec{A} \otimes_p \dots \otimes_p \vec{B}_{x_1} \otimes_p \dots \otimes_p \vec{B}_{x_n} \otimes_p \dots \otimes_p \vec{A})$$

↑
↕
↕
↕
↕

determined by Bethe equations
|↑⟩
|↓⟩
|↓⟩
|↑⟩

# Beyond rSGA



- Perturbed XXC model [CM (2023)]

$$H = \sum_{j=1}^L \sum_{a \in \{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|) + \alpha(|00\rangle\langle 00| + |22\rangle\langle 22|)_{j,j+1}$$

XXC model (integrable) [Maassarani (1997, 1999), de Leeuw et al. (2023)]

- Partially solvable model in  $W$

$$W = \text{span} \{|\psi_n\rangle\}_n, \quad |\psi_n\rangle = \sum_{1 \leq x_1 < \dots < x_n \leq L} \sum_{P \in \mathfrak{S}_n} A_n(P) e^{i \sum_{j=1}^n k_{P(j)} x_j} \cdot \text{tr}_a(\vec{A} \otimes_p \dots \otimes_p \vec{B}_{x_1} \otimes_p \dots \otimes_p \vec{B}_{x_n} \otimes_p \dots \otimes_p \vec{A})$$

$$h\vec{A} \otimes_p \vec{A} = h\vec{B} \otimes_p \vec{B} = 0$$

$$h\vec{A} \otimes_p \vec{B} = -\vec{A} \otimes_p \vec{B} + \vec{B} \otimes_p \vec{A}$$

$$h\vec{B} \otimes_p \vec{A} = \vec{A} \otimes_p \vec{B} - \vec{B} \otimes_p \vec{A}$$

$$\vec{B} = \text{diag}(b_0, b_1, b_0)\vec{A}$$

⇒ **H acts as spin-1/2 XXX model in  $W$  (integrable)**

$$H|\psi_n\rangle = \mathcal{E}_n(\{k_j\})|\psi_n\rangle, \quad \mathcal{E}_n(\{k_j\}) = \left( 2 \sum_{j=1}^n \cos k_j - \frac{n}{2} \right)$$

energy spectrum of spin-1/2 XXX

$$e^{ik_j L} = (-1)^{n-1} \prod_{l=1, l \neq j}^n \frac{e^{i(k_j+k_l)} + 1 - 2e^{ik_j}}{e^{i(k_j+k_l)} + 1 - 2e^{ik_l}} : \text{Bethe eq.}$$

# Beyond rSGA



- AKLT vs. Perturbed XXC : Two models with **partial solvability**.

- Reference state (vacuum)

Written by MPS with 2-dim bond space (the same one).

- Excitation states

AKLT  $\Rightarrow$  Violation of hidden +- order  $(Q^+ = \sum_j (-1)^j (S_j^+)^2)$

XXC  $\Rightarrow$  **No violation** of hidden +- order  $(Q^+ = \sum_j (-1)^j \text{diag}(b_0 b_1 b_0)_j)$

- Energy spectrum in  $W$

AKLT  $\Rightarrow$  Equally spacing (rSGA)

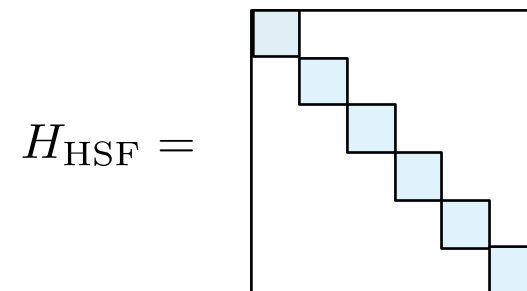
XXC  $\Rightarrow$  **Not equally spacing** (spin-1/2 XXX spectrum)

$\Rightarrow$  **Hilbert space fragmentation!**

- Dimension of  $W$

AKLT  $\Rightarrow$  Linear growth with the system size  $L$

XXC  $\Rightarrow$  **Exponential growth** with the system size  $L$



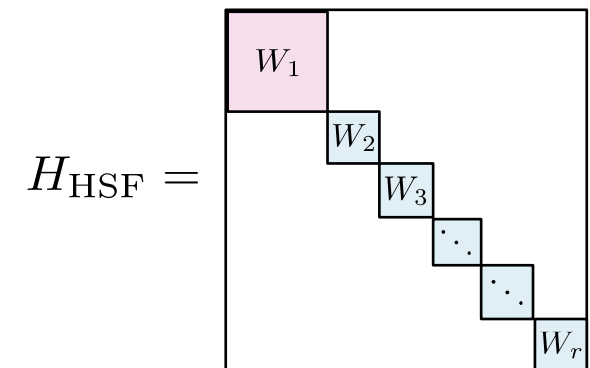
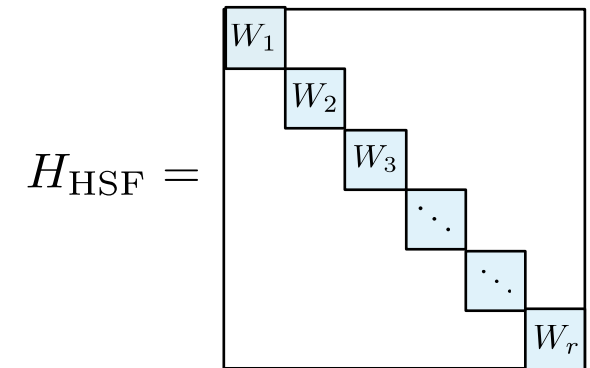
# Embedded integrable models

- **Hilbert-space fragmentation** (HSF; Krylov restricted thermalization)

[Nicolai (1976), Batista et al. (1995), Zhang et al. (1997), Batista-Ortiz (2000)]

$$\mathcal{H} = \bigoplus_{\alpha=1}^r W_{\alpha}, \quad W_{\alpha} = \text{span} \{H^{n_{\alpha}} |\psi_{\alpha}\rangle\}_{n_{\alpha}}$$

- Hilbert space fragmented into exponentially many invariant subspaces
- **Fragmented subspaces are not distinguished by obvious local symmetries of Hamiltonians.**
- Solvable subspaces are sometimes embedded (not always).



# Embedded integrable models

- Example: XXC model [Maassarani (1997, 1999), de Leeuw et al. (2023), CM (2023), Katsura-CM-Paletta-Pozsgay (2025)]

$$H = \sum_{j=1}^L \sum_{a \in \{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)$$

- Yang-Baxter solvable model
- Exhibits HSF

The interaction never changes configurations of 0 & 2 : **“Irreducible string (IS)”**

$$\begin{aligned} \text{e.g.) } H : |\underline{10211210}\rangle &\mapsto (h_{10}^{10} + h_{21}^{21} + h_{12}^{12} + h_{21}^{21} + h_{10}^{10}) |\underline{10211210}\rangle \\ &+ h_{01}^{10} |\underline{01211210}\rangle + h_{12}^{21} |\underline{10121210}\rangle + h_{21}^{12} |\underline{10212110}\rangle \quad (\text{IS} = 0220) \\ &+ h_{12}^{21} |\underline{10211120}\rangle + h_{01}^{10} |\underline{10211201}\rangle \end{aligned}$$

Original configuration

0201021111  
0120110211



020\*02\*\*\*  
0\*20\*\*02\*\*

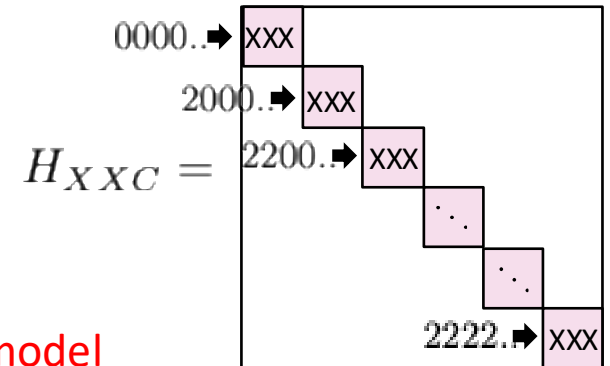
Irreducible string

02002  
02002

# Embedded integrable models

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spin-1/2 XXX model  
in every subsp.

- Yang-Baxter solvable model
- Exhibits HSF

In the subsp  $W_{up}$  of 0000... (the all up state)

$$H|_{W_{up}} = \sum_j (|01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|)_{j,j+1} \simeq H_{XXX} \Rightarrow \text{spin-1/2 model!}$$

In the subsp  $W_{alt}$  of 0202... (the alternating state)

$$H|_{W_{alt}} = \sum_j (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)_{j,j+1} \simeq H_{XXX} \Rightarrow \text{spin-1/2 model!}$$

↑  
fixed by the initial state

Frozen 0&2-configuration

⇒ The Hamiltonian effectively acts as spin-1/2 model!

# Embedded integrable models

- Example:  $XXC$  model [Maassarani (1997, 1999), de Leeuw et al. (2023), CM (2023), Katsura-CM-Paletta-Pozsgay (2025)]

$$H = \sum_{j=1}^L \sum_{a \in \{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)_{j,j+1}$$

- Violation of integrability [Katsura-CM-Paletta-Pozsgay (2025)]

Integrability is easily violated by perturbations.

**The target space can remain intact.**

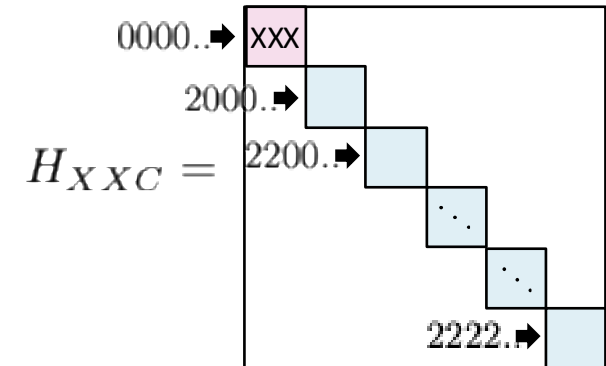
- No violation of HSF
- Acts as zero in a target space.

Ex.)  $H_{\text{pert}} = \alpha \sum_j (|02\rangle\langle 02| + |20\rangle\langle 20|)_{j,j+1}$

no violation of HSF  
acts as zero in  $W_{\text{up}}$

$$\Rightarrow H + H_{\text{pert}}|_{W_{\text{up}}} \simeq H_{XXX}$$

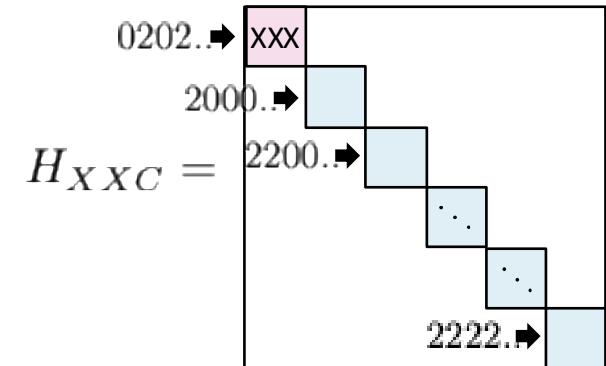
(but non-integrable in a generic subspace. )



# Embedded integrable models

- Example: XXC model [Maassarani (1997, 1999), de Leeuw et al. (2023), CM (2023), Katsura-CM-Paletta-Pozsgay (2025)]

$$H = \sum_{j=1}^L \sum_{a \in \{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)_{j,j+1}$$



- Violation of integrability [Katsura-CM-Paletta-Pozsgay (2025)]

Integrability is easily violated by perturbations.

**The target space can remain intact.**

- No violation of HSF
- Acts as zero in a target space.

Ex.)  $H_{\text{pert}} = \alpha \sum_j (|00\rangle\langle 00| + |22\rangle\langle 22|)_{j,j+1} \Rightarrow$  **Perturbed XXC model!!** (The alternating subspace remains intact.)

No violation of HSF  
Acts as zero in  $W_{\text{alt}}$

$$\Rightarrow H + H_{\text{pert}}|_{W_{\text{alt}}} \simeq H_{\text{XXX}}$$

(but non-integrable in a generic subspace.)

# Embedded integrable models

- Example:  $XXC$  model [Maassarani (1997, 1999), de Leeuw et al. (2023), CM (2023), Katsura-CM-Paletta-Pozsgay (2025)]

$$H = \sum_{j=1}^L \sum_{a \in \{0,2\}} (|a1\rangle\langle a1| + |1a\rangle\langle 1a| + |a1\rangle\langle 1a| + |1a\rangle\langle a1|)_{j,j+1}$$

- Manipulation of embedded integrability

We can choose a target subspace.

[Katsura-CM-Paletta-Pozsgay (2025)]

– Integrability-violating perturbations act as zero in a target space.

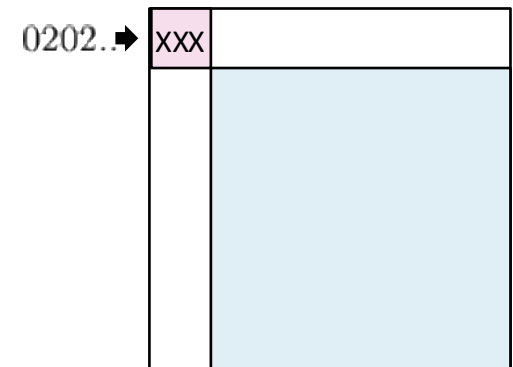
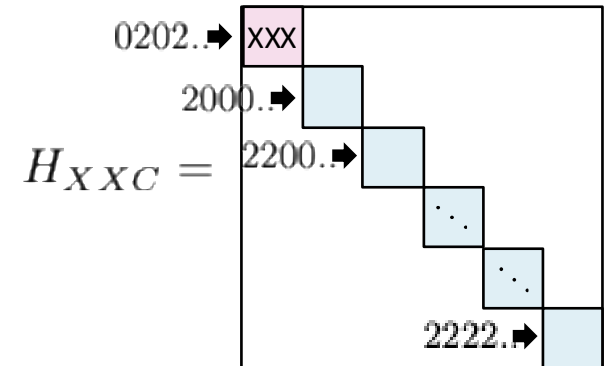
⇒ Arbitrary chosen as long as they are irrelevant in a target subspace.

⇒ HSF is not necessary for embedding integrability!

$$H_{\text{pert}} = \beta \sum_j (|00\rangle\langle 22| + |22\rangle\langle 00|)_{j,j+1}$$

Violates HSF ⇒ Connects most subspaces.

Acts as zero in  $W_{\text{alt}}$  ⇒ Keeps integrability in  $W_{\text{alt}}$ .



**Integrability embedding perturbations can be site-dependent.**

# Beyond isolated quantum systems



- Isolated quantum systems with QMBS
  - have a relatively small invariant subspace.
  - have an extra algebraic structure in the scar subspace.  
Ex.) rSGA, Yang-Baxter equation for HSF, etc.
  - are very often “partially solvable”.

**Is partial solvability only for closed quantum systems?**

**Can partial solvability be extended to open quantum systems?**

⇒ Anomalous (QMBS-like) states in open quantum systems.  
HSF in open quantum systems.

# Outline



- Thermalization & Quantum many-body scars (QMBS)
- Partially solvable isolated quantum systems
  - QMBS induced by restricted spectrum-generating algebra (rSGA)
  - QMBS induced by Hilbert space fragmentation (HSF)
- Partially solvable open quantum systems
- Concluding remarks

# Partially solvable open quantum systems



- Lindblad master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)), \quad \mathcal{L}(\rho) = -i[H, \rho] + \sum_{\mu} \varepsilon_{\mu} \mathcal{D}_{\mu}(\rho)$$

Steady state  
as the fixed point of  $\mathcal{L}$

$$\mathcal{D}_{\mu}(\rho) = \underbrace{2A_{\mu}\rho A_{\mu}^{\dagger}}_{\text{Quantum jump operator}} - \{A_{\mu}^{\dagger}A_{\mu}, \rho\} : \text{Dissipation terms}$$

- The most general CPTP map for Markovian time evolution.

- Some partially solvable examples:

Fully solvable  $\Rightarrow$  Free-fermionization, Bethe-ansatz solvability, Triangularization, etc.

[Prosen (2008), Medvedyeva et al. (2016), Buca et al. (2020)]

Partially solvable  $\Rightarrow$  Bulk solvability + “a good choice of dissipators” [Prosen (2013), Karevski et al. (2013)]

Partial solvability induced by bulk partial solvability?

Can partial solvability be robust against the boundary dissipators?

# HSF-induced solvable eigenmodes

- Boundary dissipative quantum system

- Lindblad master equation

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{\mu} \gamma_{\mu} \mathcal{D}_{\mu}(\rho)$$

$$\mathcal{D}_{\mu}(\rho) = 2A_{\mu}\rho A_{\mu}^{\dagger} - \{A_{\mu}^{\dagger}A_{\mu}, \rho\}$$

- Isomorphism to doubled Hilbert space

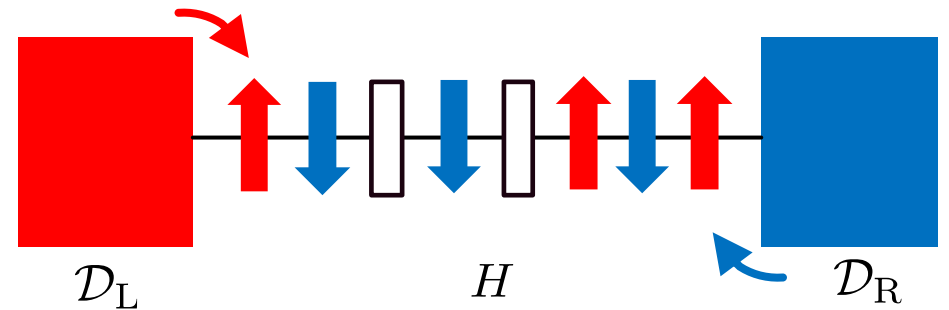
$$\rho = \sum_{m,n} \rho_{m,n} |m\rangle\langle n| \mapsto \sum_{m,n} \rho_{m,n} |m\rangle \otimes |n\rangle = |\rho\rangle\rangle$$

density matrix  $\in \mathcal{H}^{\dagger} \otimes \mathcal{H}$       vector in a doubled Hilbert space  $\in \mathcal{H}^* \otimes \mathcal{H}$

- Evolution of the density matrix

$$\frac{d}{dt} |\rho(t)\rangle\rangle = -i\tilde{H} |\rho(t)\rangle\rangle$$

$$\tilde{H} = \underbrace{H \otimes \mathbf{1} - \mathbf{1} \otimes {}^tH}_{\text{bulk Hamiltonian}} + \underbrace{i \sum_{\alpha} \gamma_{\alpha} \left( (A_{\alpha} \otimes A_{\alpha}^*) - \frac{1}{2} (A_{\alpha}^{\dagger} A_{\alpha} \otimes \mathbf{1} + \mathbf{1} \otimes {}^t A_{\alpha} A_{\alpha}^*) \right)}_{\text{boundary dissipation}} \quad : \text{Non-Hermitian effective Hamiltonian}$$



**Treated as an isolated quantum system with a non-Hermitian Hamiltonian!**

**Can  $\tilde{H}$  be an integrable Hamiltonian in a certain subspace?**

# HSF-induced solvable eigenmodes

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]

- XXC model exhibits HSF by configuration of 0 & 2.

⇒ Integrable spin-1 model.

**Integrability is embedded in a selected sector by perturbations.**

**Partial solvability intact by site-dependent perturbations.**

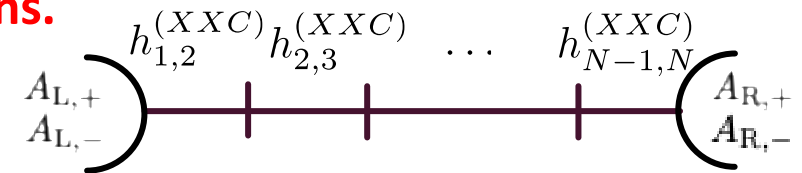
- Boundary dissipators

$$A_{L,+} = (S_1^+)^2, A_{L,-} = (S_1^-)^2, A_{R,+} = (S_N^+)^2, A_{R,-} = (S_N^-)^2$$

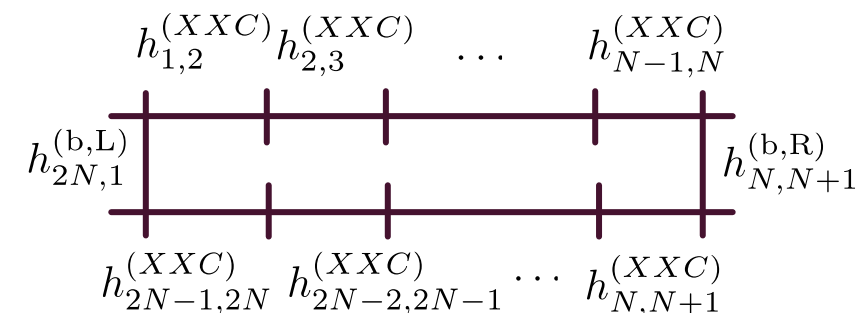
- Effective non-Hermitian Hamiltonian

$$\tilde{H}_{XXC} = \sum_{n=1}^{N-1} h_{n,n+1}^{(XXC)} + h_{N,N+1}^{(b,R)} + \sum_{n=N+1}^{2N} h_{n,n+1}^{(XXC)} + h_{2N,1}^{(b,L)}$$

⇒ **Two XXC chains coupled at the boundaries.**



⇓ thermofield double



# HSF-induced solvable eigenmodes

- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]

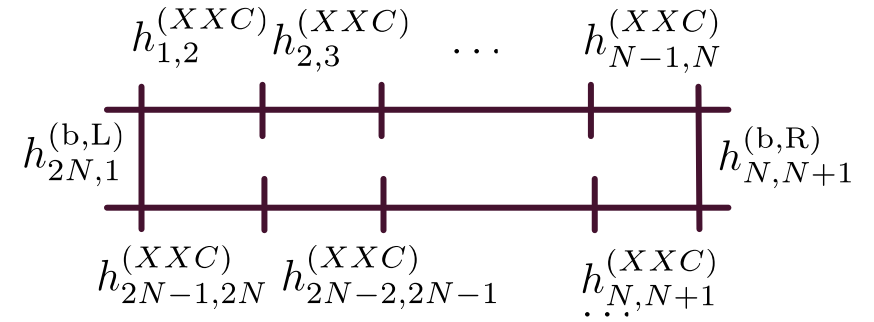
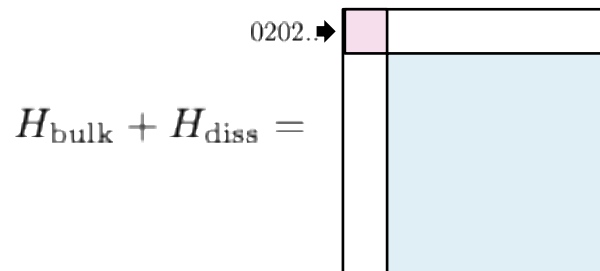
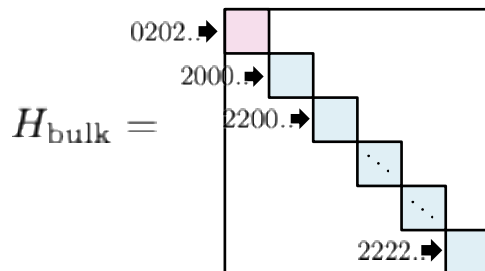
- Effective non-Hermitian Hamiltonian

$$\tilde{H}_{XXC} = \sum_{n=1}^{N-1} h_{n,n+1}^{(XXC)} + h_{N,N+1}^{(b,R)} + \sum_{n=N+1}^{2N} h_{n,n+1}^{(XXC)} + h_{2N,1}^{(b,L)}$$

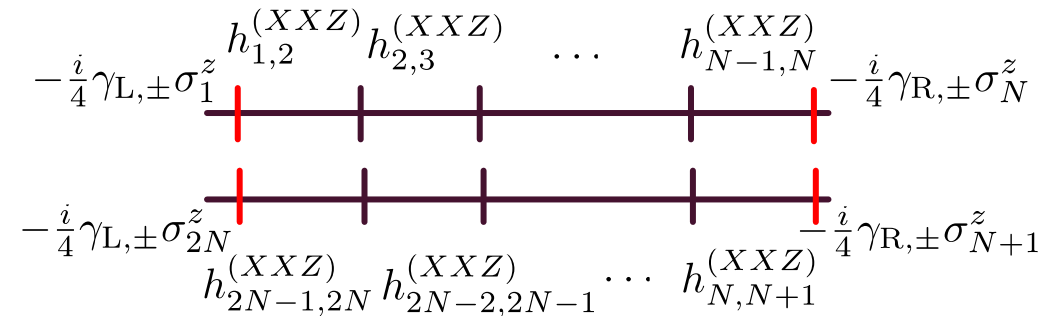
HSF  $\Rightarrow$  configuration of 0 & 2 are frozen.

$$h^{(b,\alpha)} = i\gamma_{\alpha,+} (|00\rangle\langle 22| - \frac{1}{2}(|2\rangle\langle 2| \otimes \mathbf{1} + \mathbf{1} \otimes |2\rangle\langle 2|)) + i\gamma_{\alpha,-} (|22\rangle\langle 00| - \frac{1}{2}(|0\rangle\langle 0| \otimes \mathbf{1} + \mathbf{1} \otimes |0\rangle\langle 0|))$$

Boundary terms does not violate the alternating subspace!



$\Downarrow \mathcal{P}_{\text{alt}} \mathcal{H} \setminus \{0, 2\}^{\otimes N}$



$\Rightarrow$  Two decoupled XXZ chains with boundary magnetic fields (integrable).

# HSF-induced solvable eigenmodes



- Example: XXC Hamiltonian coupled to boundary dissipators [CM-Tsuji (2024)]

- Eigenstates of effective Hamiltonian

**Effect of boundary dissipators remain finite in the solvable steady state!**

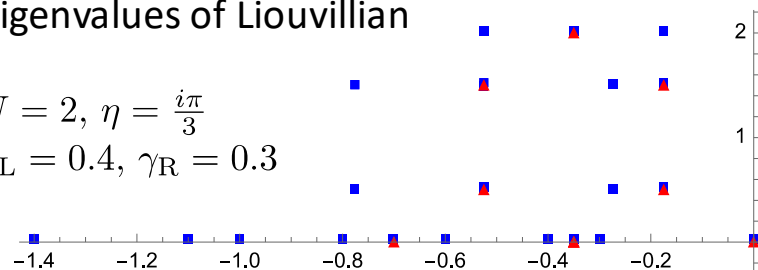
$$\tilde{H} \xrightarrow{\mathcal{P}_{\text{alt}} \mathcal{H} \setminus \{0,2\}^{\otimes N}} H_{XXZ}^{(+)}(\gamma_L, \gamma_R) \otimes \mathbf{1} - \mathbf{1} \otimes H_{XXZ}^{(-)}(\gamma_L, \gamma_R)$$

Ex.) Dark states [Diehl et al. (2008), Kraus et al. (2008), CM-Tsuji (2024)]

$$H_{XXZ}^{(\pm)}(\gamma_L, \gamma_R) = \frac{1}{2} \sum_{x=1}^{L-1} (\sigma_x^+ \sigma_{x+1}^- + \sigma_x^- \sigma_{x+1}^+ + \frac{1}{2} \cosh \eta \cdot \sigma_x^z \sigma_{x+1}^z) \pm \frac{i}{4} \gamma_L \sigma_1^z \pm \frac{i}{4} \gamma_R \sigma_N^z$$

Eigenvalues of Liouvillian

$N = 2, \eta = \frac{i\pi}{3}$   
 $\gamma_L = 0.4, \gamma_R = 0.3$



- full eigenvalues of  $\mathcal{L}_{XXC}$
- ▲ eigenvalues of  $\mathcal{L}_{XXC}$  in the solvable subspace

The full Liouvillian has degenerate steady states including the integrable steady state and the fully polarized state.

**Anomalous “integrable” behaviors are expected for the initial state overlapping with the solvable subspace.**

# rSGA-induced solvable eigenmodes

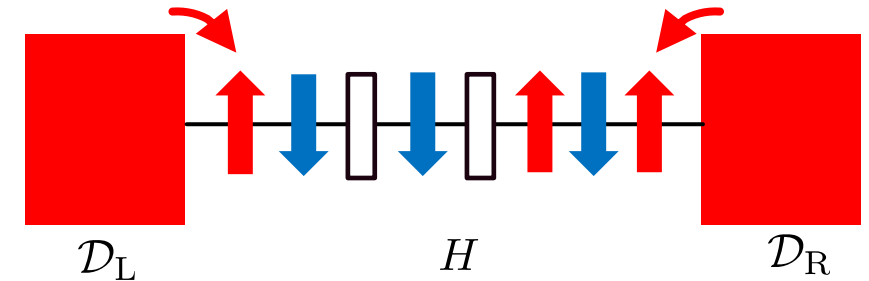


- AKLT model coupled to boundary dissipators

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{\mu} \gamma_{\mu} \mathcal{D}_{\mu}(\rho)$$

$$\mathcal{D}_{\mu}(\rho) = 2A_{\mu}\rho A_{\mu}^{\dagger} - \{A_{\mu}^{\dagger}A_{\mu}, \rho\}$$

[CM-Tsuji (2024)]



- Bulk Hamiltonian with rSGA

$$[H, Q^+] - 2\mathcal{E}Q^+|_W = 0, \quad W = \text{span}\{\underbrace{(Q^+)^n|\psi_0\rangle}_n\}, \quad Q^+ = \sum_{j=1}(-1)^j(S_j^+)^2$$

eigenstates  $\Rightarrow$  commutes with H : quasiparticle creation operator

- Quasiparticle baths at the edges

$$A_L = (S_1^+)^2, \quad A_R = (S_L^+)^2 \Rightarrow \text{Doping quasiparticles from both edges.}$$

Boundary dissipators usually violate partial solvability.

- Partial-solvability non-violating dissipators

$$\mathcal{D}_{\mu}((Q^+)^n|\psi_0\rangle\langle\psi_0|(Q^+)^n) = 0, \quad \forall \mu. \Rightarrow \text{Dissipators are irrelevant on quantum scars (dark states).}$$

[Diehl et al. (2008), Kraus et al. (2008), CM-Tsuji (2024)]

# rSGA-induced solvable eigenmodes



- AKLT model coupled to boundary dissipators

[CM-Tsuji (2024)]

- Quantum scars of bulk AKLT model

Eigenstates of the AKLT model  $\Rightarrow$  commutes with H

$$[H, (Q^+)^n |\psi_0\rangle\langle\psi_0| (Q^{+\dagger})^n] = 0$$

Dark states  $\Rightarrow$  Dissipators are irrelevant for quantum scars.

$$\mathcal{D}_\mu((Q^+)^n |\psi_0\rangle\langle\psi_0| (Q^{+\dagger})^n) = 0, \quad \forall \mu.$$

- Any diagonal ensemble becomes the steady state of boundary-dissipative AKLT.

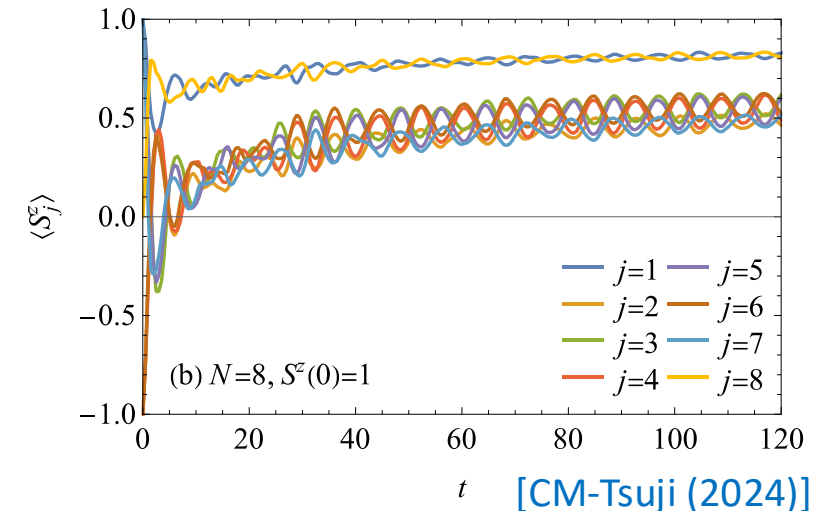
$$\mathcal{L}\left(\sum_n p_{nn} (Q^+)^n |\psi_0\rangle\langle\psi_0| (Q^{+\dagger})^n\right) = 0, \quad \sum_n p_{nn} = 1, p_{nn} > 0, \quad \forall n$$

- Persistent oscillations

$$|\Psi(t=0)\rangle = \sum_n a_n (Q^+)^n |\psi_0\rangle$$

$\Rightarrow$  Long-lived oscillations are observed!

$$\Rightarrow \lim_{t \rightarrow \infty} \langle O(t) \rangle = \sum_{n \leq m} \underline{2 \cos((m-n)\mathcal{E}t)} a_m a_n \operatorname{Re} O_{nm}$$



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# Take-home messages



- Quantum many-body scars often originate from partial solvability.
- Two mechanisms of partial solvability:
  - restricted spectrum-generating algebra
  - Hilbert space fragmentation
- Partial solvability survives even in open systems.

# Future works



- Quantum scars

- relatively small entanglement entropy
- long-lived information of the initial state

⇒ **Can quantum scars inherit the ground state properties?**

Ex.) Quantum phase transition such as SPT phase [CM-Quella-Tsuji (2025)]

- Hilbert space fragmentation

- embedded integrability robust by site-dependent perturbations

⇒ **Can HSF-induced non-thermal subspaces selectively host physical properties?**

[Work in progress with Imai & Tsuji]