Revised: October 26, 2023

Corrections, Addenda, and Errata to "Physics and Mathematics of Quantum Many-Body Systems"

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Correction to Section 3.3

There is an essential flaw in the discussion of the explicit behavior of the energy gap. I thank Haruki Watanabe for pointing this out.

The formula for the energy difference between the ground state and the first excited state of the one-dimensional quantum Ising model that appears at the bottom of page 58 is incorrect. The correct formula (for $\lambda \ll 1$) is

$$E_{\mathrm{1st}} - E_{\mathrm{GS}} \simeq \frac{1}{2}(2\lambda)^L.$$

The derivation can be found in the following paper.[1]
G. G. Cabrera and R. Jullien, *Role of boundary conditions in the finite-size Ising model*, Phys. Rev. **B** 35, 7062 (1987).

I must also note that Problem 3.3.a and its solution discuss an approximation (which I incorrectly regarded to be controlled for $\lambda \ll 1$) for deriving the *wrong* estimate $E_{\mathrm{1st}} - E_{\mathrm{GS}} \simeq 2\lambda^L$. Since there seems to be no mistake in the calculation presented in the solution (p.498), we must conclude that the basic strategy of the approximation is flawed. The idea for the approximation is that one can characterize the ground state and the first excited state by only considering the subspace that consists of the two classical ground states and states with a single kink. It is even claimed in footnote 12 that this treatment is justified by the cluster expansion method, but, in retrospect, I do not find this claim convincing. It must be the case that one must consider other states, namely, states with multiple kinks, to fully recover the correct behavior of the energy difference $E_{\mathrm{1st}} - E_{\mathrm{GS}}$.

In conclusion, I must take back Problem 3.3.a and its solution (which may be one of the longest solutions in the book). The above flaw does not affect the general discussion that starts with **Exact ground state versus "physical ground states"**.

Addendum to Section 2.1

In Problem 2.1.g (page 20), I asked the reader to derive explicit matrix forms of the π-rotation operators about the three axes. Since I was not able to devise any compact

[1] See (52) in p.7067. Here N, γ, and $m^{\mathrm{FE}}(N,\gamma)$ correspond to our L, 2λ, and $4(E_{\mathrm{1st}} - E_{\mathrm{GS}})$, respectively.

direct derivation, I suggested reducing the problem to that of a collection of $S=1/2$ spins. After the publication of the book, however, I learned the following elegant derivation from Keisuke Murota. I believe this one is much better than the solution I gave in the book.

A better solution to Problem 2.1.g

We shall show below that $\langle\psi^\sigma|\hat{u}_1|\psi^\tau\rangle=(-i)^{2S}\,\delta_{\sigma,-\tau}$. The relation $\langle\psi^\sigma|\hat{u}_3|\psi^\tau\rangle=e^{-i\pi\sigma}\,\delta_{\sigma,\tau}$ is obvious. Then the relation $\langle\psi^\sigma|\hat{u}_2|\psi^\tau\rangle=(-1)^{S+\sigma}\,\delta_{\sigma,-\tau}$ follows from $\hat{u}_2=\hat{u}_3\hat{u}_1$.

From (2.1.21), we see that

$$\hat{S}^{(\alpha)}\hat{u}_1=-\hat{u}_1\hat{S}^{(\alpha)} \tag{AE.1}$$

for $\alpha=2,3$. Since

$$\hat{S}^{(3)}\hat{u}_1|\psi^\sigma\rangle=-\hat{u}_1\hat{S}^{(3)}|\psi^\sigma\rangle=-\sigma\,\hat{u}_1|\psi^\sigma\rangle, \tag{AE.2}$$

we find that

$$\hat{u}_1=\sum_\sigma\lambda_\sigma|\psi^{-\sigma}\rangle\langle\psi^\sigma|, \tag{AE.3}$$

with some λ_σ.

Note that (AE.1) also implies $\hat{S}^+\hat{u}_1=\hat{u}_1\hat{S}^-$. By using (2.1.3) and (AE.3), we find

$$\hat{S}^+\hat{u}_1|\psi^\sigma\rangle=\lambda_\sigma\,\hat{S}^+|\psi^{-\sigma}\rangle=\lambda_\sigma\,\sqrt{S(S+1)+\sigma(-\sigma+1)}\,|\psi^{-\sigma+1}\rangle, \tag{AE.4}$$

and

$$\begin{aligned}\hat{u}_1\hat{S}^-|\psi^\sigma\rangle&=\sqrt{S(S+1)-\sigma(\sigma-1)}\,\hat{u}_1|\psi^{\sigma-1}\rangle\\&=\lambda_{\sigma-1}\,\sqrt{S(S+1)-\sigma(\sigma-1)}\,|\psi^{-\sigma+1}\rangle.\end{aligned} \tag{AE.5}$$

Since these two must be identical, we find $\lambda_\sigma=\lambda_{\sigma-1}$, and hence

$$\hat{u}_1=\lambda\,\hat{R}, \tag{AE.6}$$

for some $\lambda\in\mathbb{C}$, where

$$\hat{R}=\sum_\sigma|\psi^{-\sigma}\rangle\langle\psi^\sigma|. \tag{AE.7}$$

We only need to determine λ. Let $|\xi^S\rangle$ be the unique state such that $\hat{S}^{(1)}|\xi^S\rangle=S|\xi^S\rangle$, which obviously satisfies $\hat{u}_1|\xi^S\rangle=e^{-i\pi S}|\xi^S\rangle$. We claim that $\hat{R}|\xi^S\rangle=|\xi^S\rangle$, which shows the desired result $\lambda=e^{-i\pi S}=(-i)^{2S}$.

To see the claim we first note that $\hat{R}|\xi^S\rangle=\pm|\xi^S\rangle$ because $[\hat{R},\hat{u}_1]=0$ and $\hat{R}^2=\hat{1}$. Note that $|\xi^S\rangle$ is obtained by maximizing $\langle\xi|\hat{S}^{(1)}|\xi\rangle=\langle\xi|(\hat{S}^++\hat{S}^-)|\xi\rangle/2$ over all normalized $|\xi\rangle=\sum_\sigma c_\sigma|\psi^\sigma\rangle$. It is obvious that the maximum is attained when $c_\sigma\ge 0$ for all σ, and hence it is only possible that $\hat{R}|\xi^S\rangle=|\xi^S\rangle$. (This is essentially the same as using the Perron-Frobenius theorem.)

Addendum to Section 8.3.5

In footnote 50 (page 278), I mentioned a generalization of Theorem 8.6 that is parallel to Theorem 8.7. Recently we found a new proof of such a generalized theorem. Interestingly the proof makes an essential use of the index introduced in the study of symmetry protected topological phase.

Yoshiko Ogata, Yuji Tachikawa, and Hal Tasaki, *General Lieb-Schultz-Mattis type theorems for quantum spin chains*, Comm. Math. Phys. **385**, 79–99 (2021).
https://arxiv.org/abs/2004.06458

Addendum to Section 11.3.2

Theorem 11.13, which is based on [34], is not correct as it is. When $(\tilde{\Lambda}, \tilde{\mathscr{B}})$ is not bipartite, the condition of biconnectedness does not guarantee the existence of ferromagnetism. See footnote in page 4338 of [35].

Errata

- p. vii, 6th line from the bottom:
  detal → detail
- p. 30, 3rd line below (2.3.28):
  identiccal → identical
- p. 37, 4th line:
  in such way → in such a way
- p. 40, 3rd and 4th lines in the Proof of Theorem 2.2:
  "that the whole spectrum of $\hat{H}$ is contained in the subspace $\mathscr{H}_0$" should better be "that, for any eigenvalue of $\hat{H}$, there is at least one corresponding eigenstate in $\mathscr{H}_0$"
- p. 54, below (3.2.9):
  $\sum_{x\in\Lambda_L}\langle\sigma_o\sigma_x\rangle_{\beta,\infty}$ should be $\sum_{x\in\mathbb{Z}^d}\langle\sigma_o\sigma_x\rangle_{\beta,\infty}$, i.e., change Λ_L to $\mathbb{Z}^d$.
- p. 193, (7.2.7):
  the minus sign on the right-hand side should be removed.
- p. 212, 8th line:
  proposal → proposed
- p. 243, 2nd line of (8.2.16):
  $\sum_{x=1}^{L} \quad \to \quad \sum_{x=1}^{L-1}$
- p. 257, 12th line:
  the map $x \to L - x \quad \to \quad$ the map $x \to L + 1 - x$

- p. 263, 18th and 19th lines:
  $x \to 1-x \quad \to \quad x \to -1-x$
- p. 265, footnote 37:
  This is not an erratum, but it is better to add the following sentence to the footnote. "Recalling that A^σ, $\tilde{\mathsf{A}}^\sigma$ and B^σ are independent of L, we see that η_L/L is independent of L."
- p. 295, the first paragraph:
  The assertion about the uniqueness of the infinite volume ground state is incorrect. In fact, it is proved in [14] that the toric code model on $\mathbb{Z}^2$ has exactly four ground states. One of them is the frustration free ground state obtained as the unique infinite volume limit of the ground states of finite systems. The remaining three ground states are not frustration free, and are characterized by the presence of an anyon. See [14] for details.
- p. 212, 15th line:
  the the $\to$ the
- p. 265:
  The logic on this page is incomplete. Theorem 7.6, as is stated, does not
- p. 316, the line below (9.2.37):
  know $\to$ known
- p. 409, right above Theorem 11.15:
  Mieleke's [36, 38] $\to$ Mielke's [36, 38]
- p. 437, 2 lines above (11.4.70):
  The inequality $\hat{P}_0\hat{h}_{\text{eff}}\hat{P}_0 \geq -2s' = -2s\nu^4/(1+\nu^2)$ should be understood as the inequality in a suitable space.
- p. 437, (11.4.74):
  The inequality should be understood as inequality in a suitable space.
- p. 480, 2nd line from the bottom:
  if and only if $\hat{A}$ is $\to$ if and only if A is
- p. 481, 10th line from the bottom:
  $2|\Lambda|S+1$ dimensional $\to$ $(2S+1)^{|\Lambda|}$ dimensional
- p. 481, 3rd line from the bottom:
  write is as $\to$ write it as
- p. 497, (S.19), 1st and 2nd lines:
  $e^{-iM\varphi/2} \;\to\; e^{-iM\varphi}$
- p. 508, (S.77):
  The product should be from $x=1$ to $L-1$.