Hierarchical Lorentz Mirror Model

Normal Transport and a Universal 2/3 Mean-Variance Law

Raphael Lefevere and Hal Tasaki

webinar @ YouTube: 2026/2/3

the image generated from the original picture of Raphael by ChatGPT 5.2 Pro

IMPORTANT NOTE

In the video, I discuss our "prediction" that, in the 2D original (non-hierarchical) Lorentz model, the mean conductance grows proportionally to logL and that the variance-to-mean ratio converges to 2/3. I even challenge numerical pros to confirm this.

However, immediately (or even instantaneously) after the first version of our paper appeared in arXiv, my friend, Hosho Katsura, pointed out to me that it's already done! (Hosho is not only a great researcher but also has such a super-human skill.)

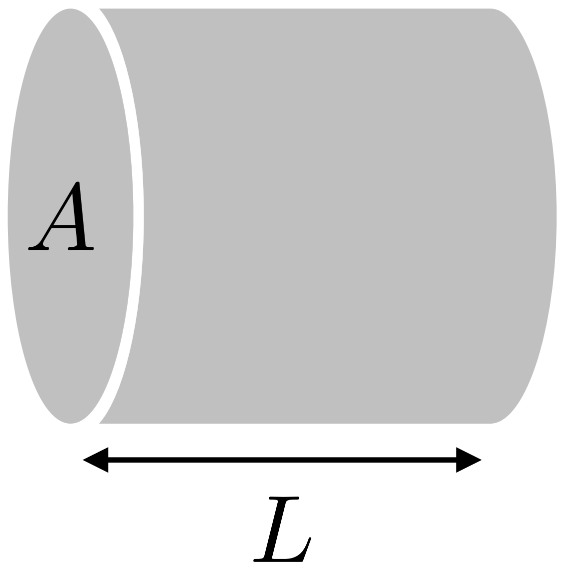
In [19], Nahum, Serna, Somoza, and Ortuno studied the model called the CPLC (compactly packed loop model with crossings), which includes the Lorenz mirror model as a special case ($q = 1/2$ and $p = 1/3$ (but all the models with $p > 0$ belong to a single universality class)), in 2D. They derived, among other things, the logL growth and the 2/3-law from a field-theoretic RG and confirmed them by a large-scale Monte Carlo simulation with L up to 10^6!! (Note: the !! expresses my astonishment, not double-factorial.) So we learned, with pleasure, that our hierarchical model provided the correct "retrodiction".

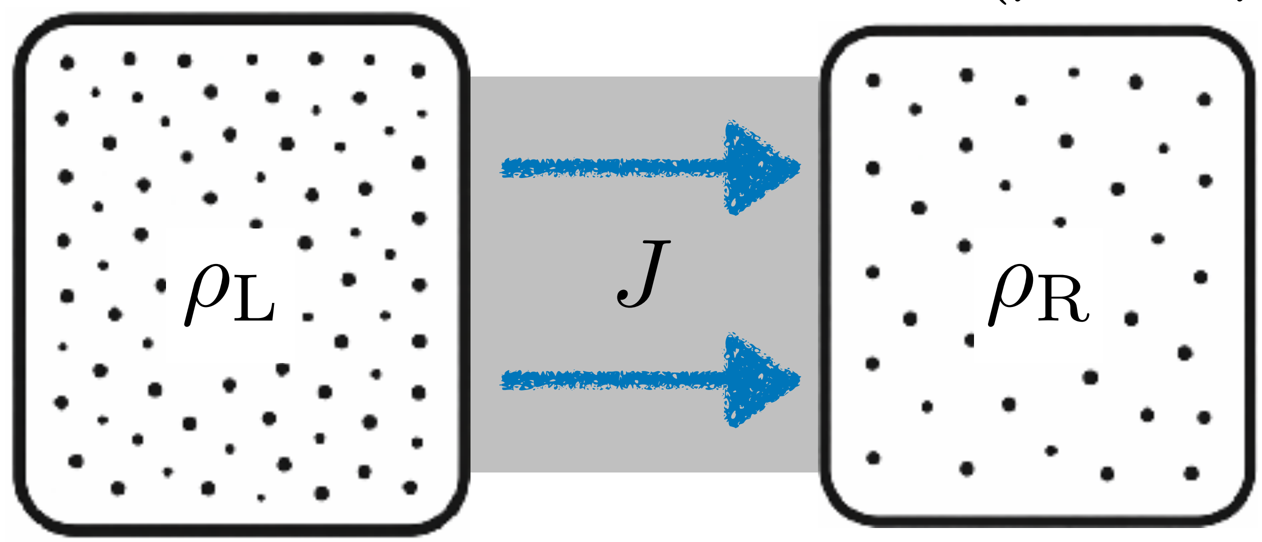
This point is taken into account in the second (and later) version of our arXiv post, but not in this video.

[19] A. Nahum, P. Serna, A. M. Somoza, and M. Ortuno, "Loop models with crossings", Phys. Rev. B 87, 184204 (2013).
https://arxiv.org/abs/1303.2342

normal transport

$$J = \mathcal{C}\,(\rho_{\mathrm{L}} - \rho_{\mathrm{R}})$$





conductance $\quad \mathcal{C} \propto \dfrac{A}{L} \quad$ cross-section / length

follows from diffusive (Ohm's, Fick's, or Fourier's) laws

do we get normal transport from fully deterministic (and non-chaotic) dynamics in a quenched random environment?

Lorentz gas, especially its discrete version, the Lorentz mirror model

Lorentz 1905, Ruijgrok, Cohen 1988

Lorentz mirror model

Hierarchical model

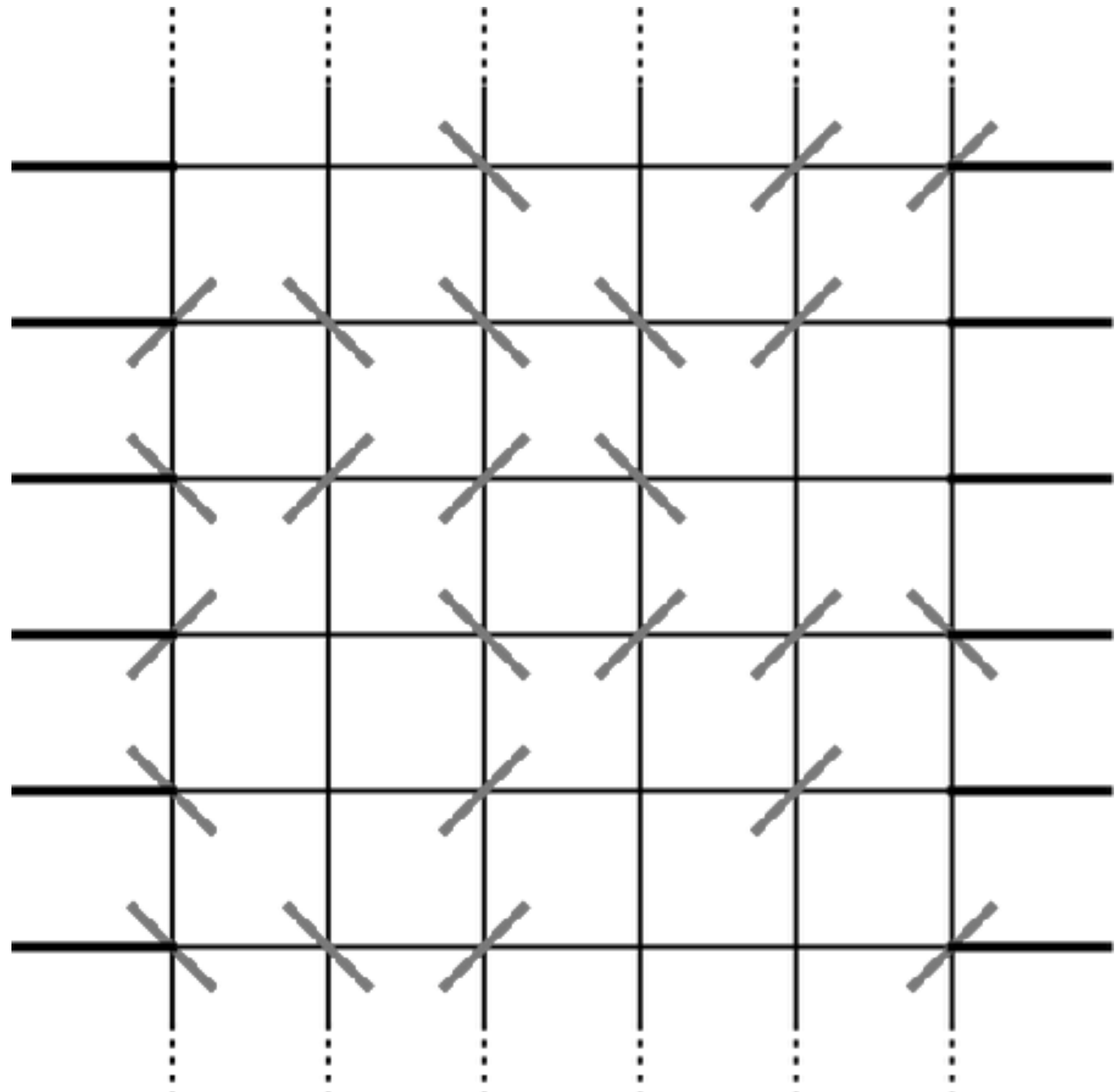
- definition and recursion relation
- main theorem on normal transport
- Gaussian closure and the "$2/3$ law"

Lorentz mirror model in $d = 3$: numerics

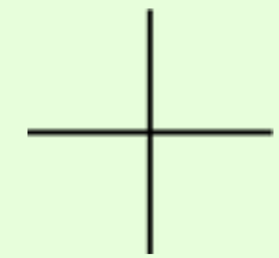
summary

appendix: brief ideas of the proof

Lorentz mirror model (d=2)



with probability 1/3



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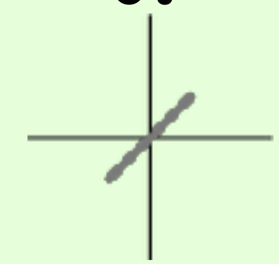

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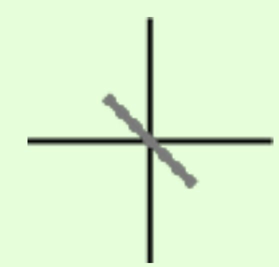


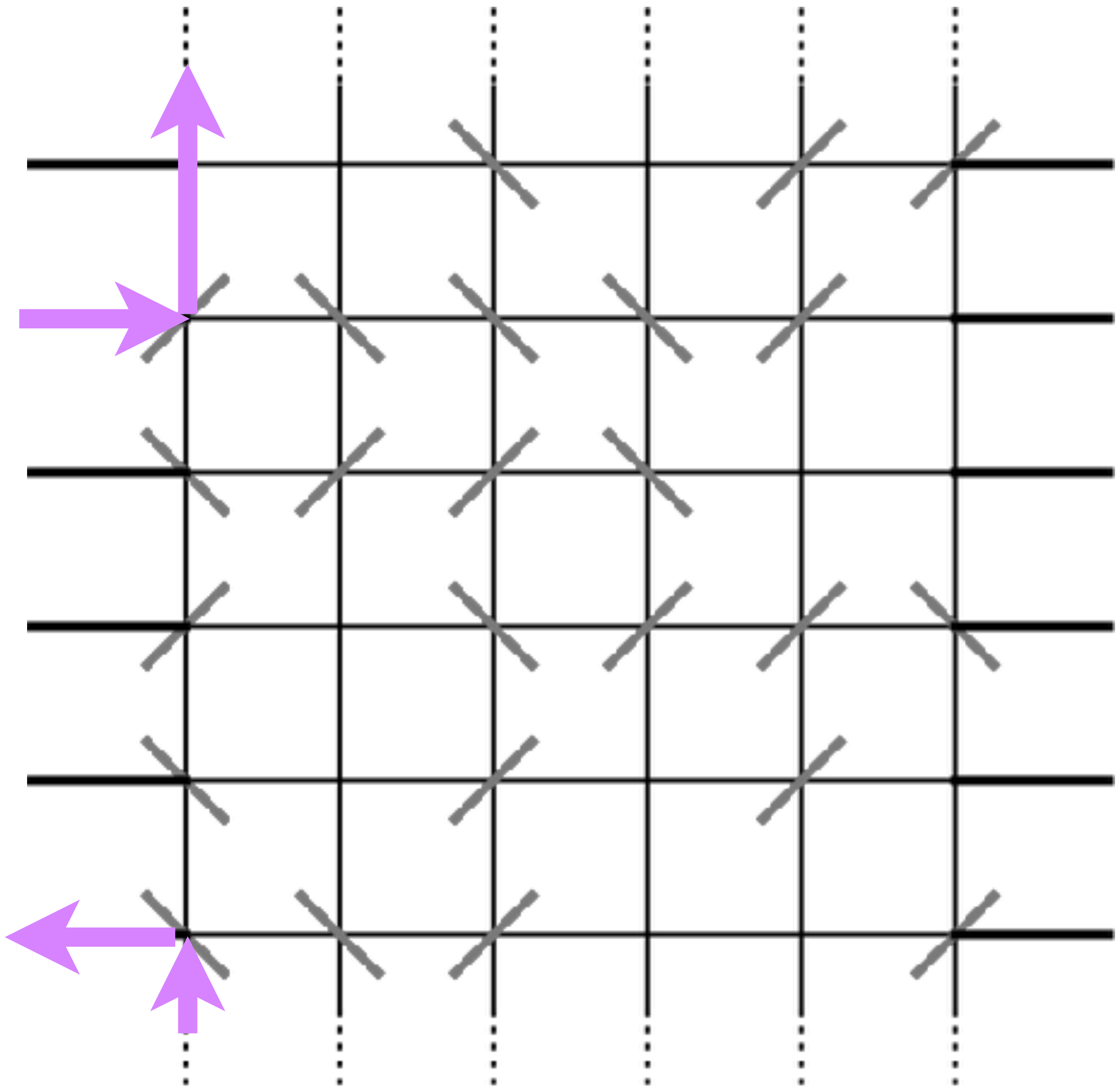
$L \times L$ square lattice

horizontal: open b.c., vertical: periodic b.c

add L external edges on the left and the right

place mirrors randomly

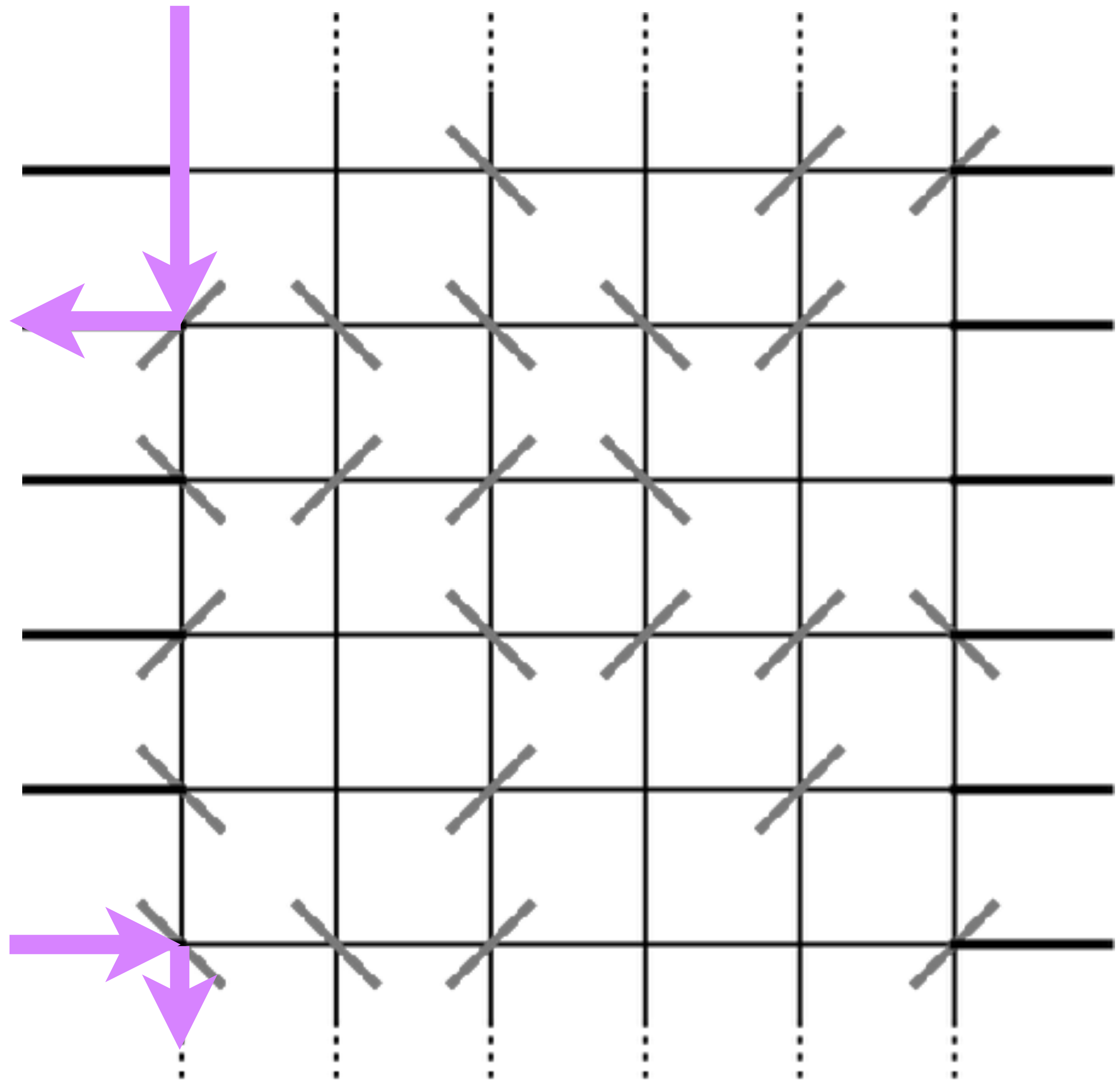
Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

illustration by John Tenniel

Lorentz mirror model (d=2)

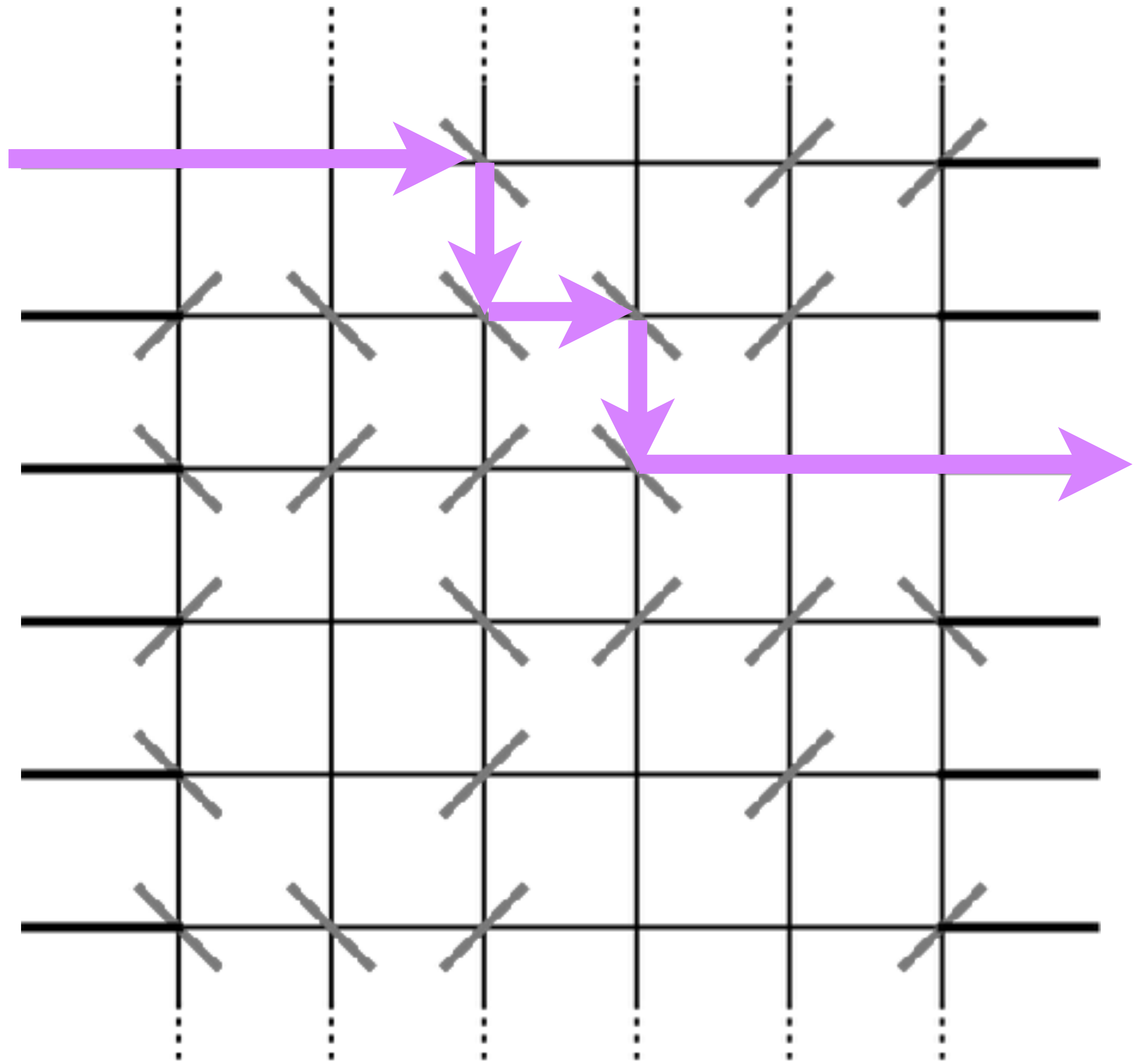


a particle (or light ray) is reflected by mirrors

trajectory is reversible

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

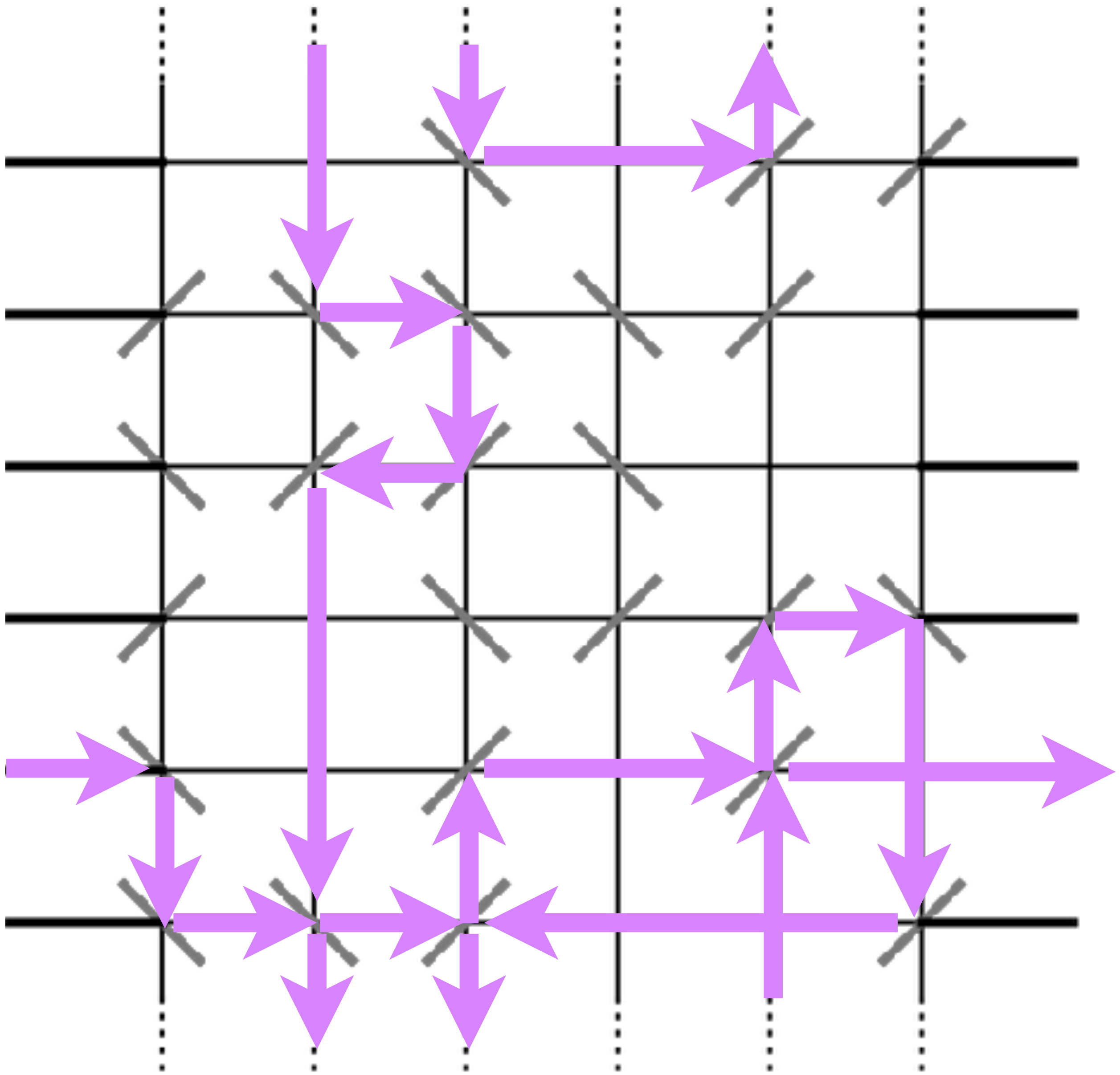
trajectory is reversible

looks like a random walk

non-backtracking random walk

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

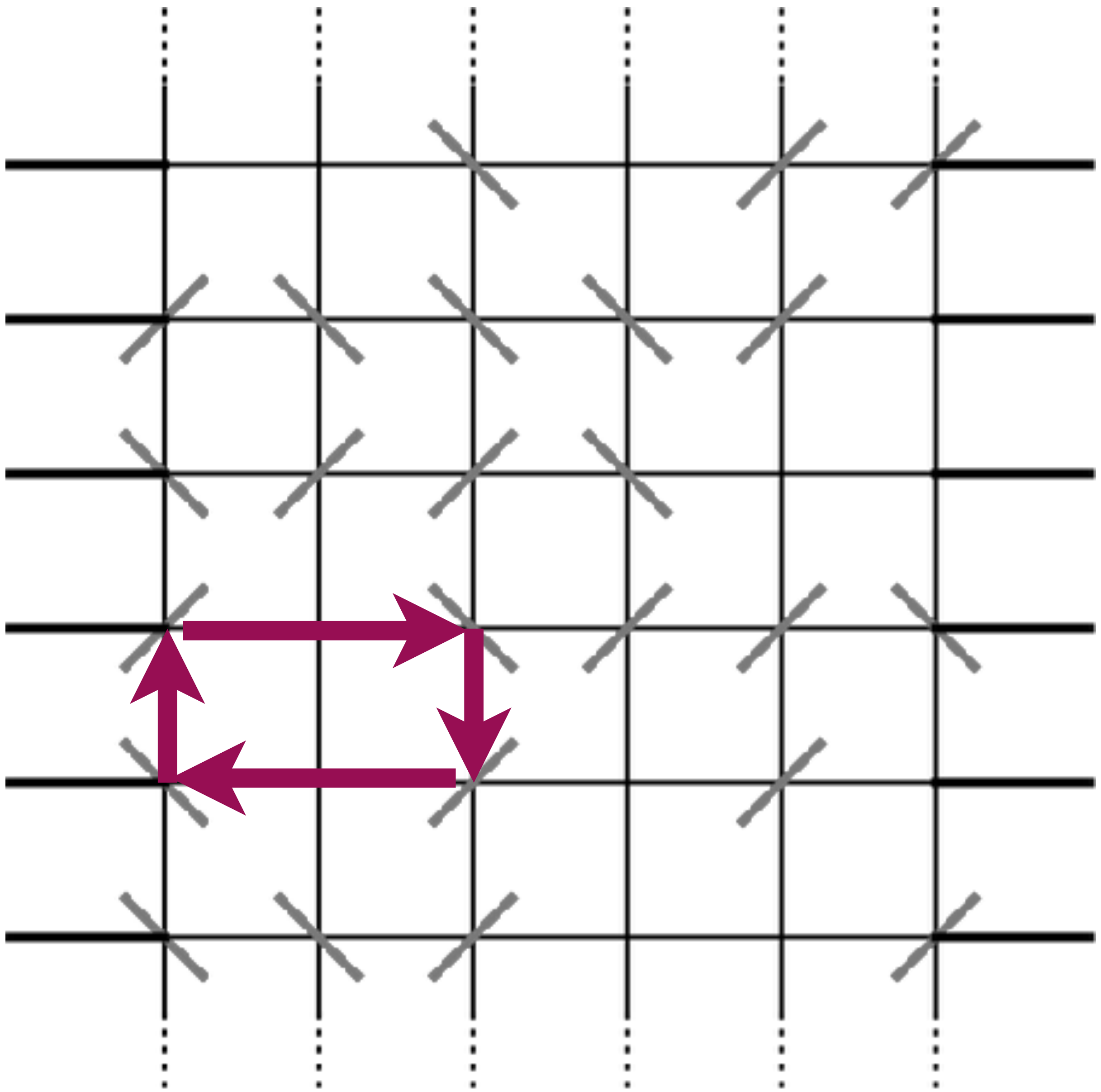
trajectory is reversible

looks like a random walk

but has strong memory!

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

trajectory is reversible

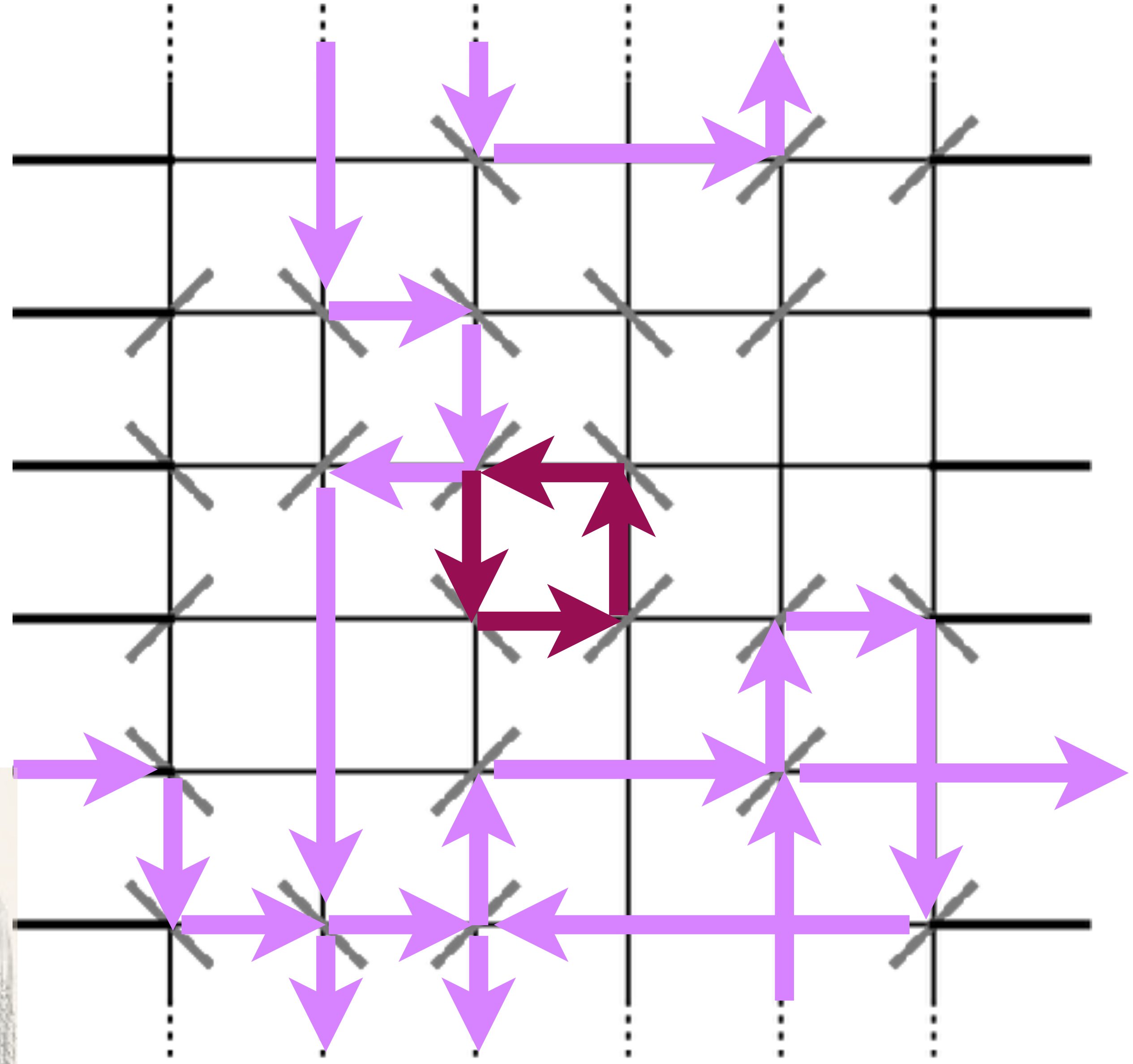
looks like a random walk

but has strong memory!

many loops!

illustration by John Tenniel

Lorentz mirror model (d=2)

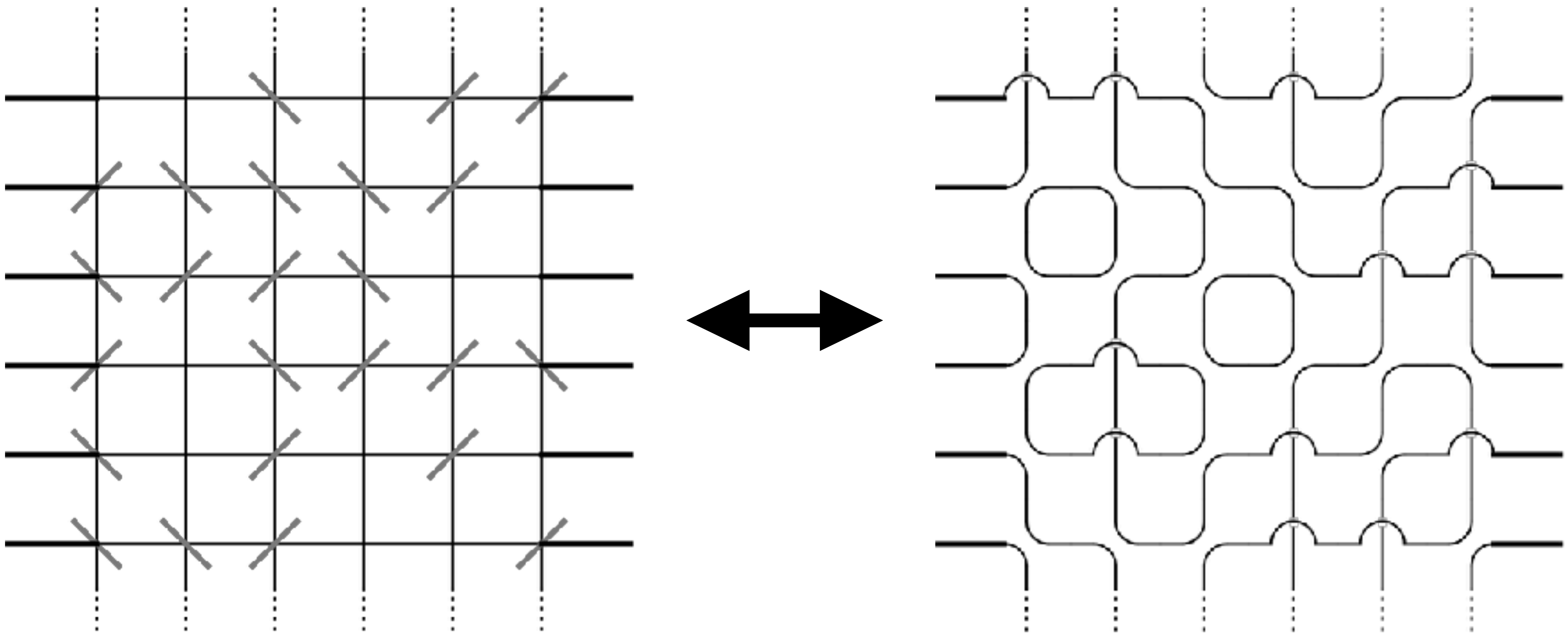


although there are many loops inside, a particle injected through an external edge must exit through another external edge (either on left or right)

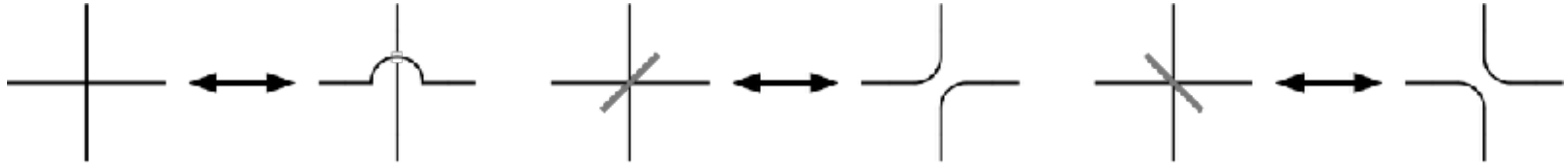
no traps!!

illustration by John Tenniel

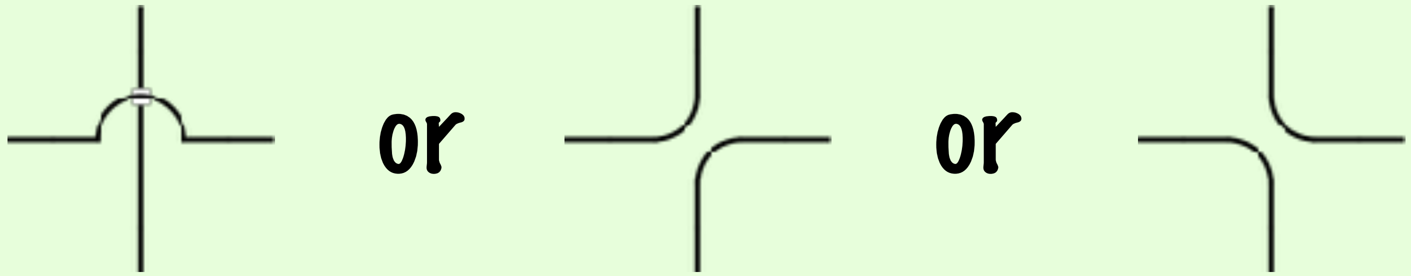
from mirrors to local pairings



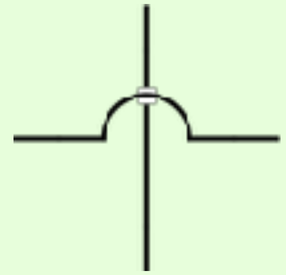
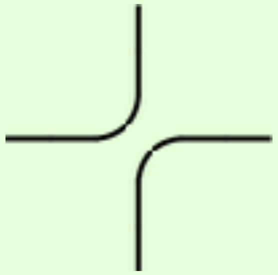
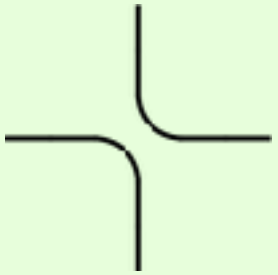
mirrors (and their absence) are equivalent to local pairings

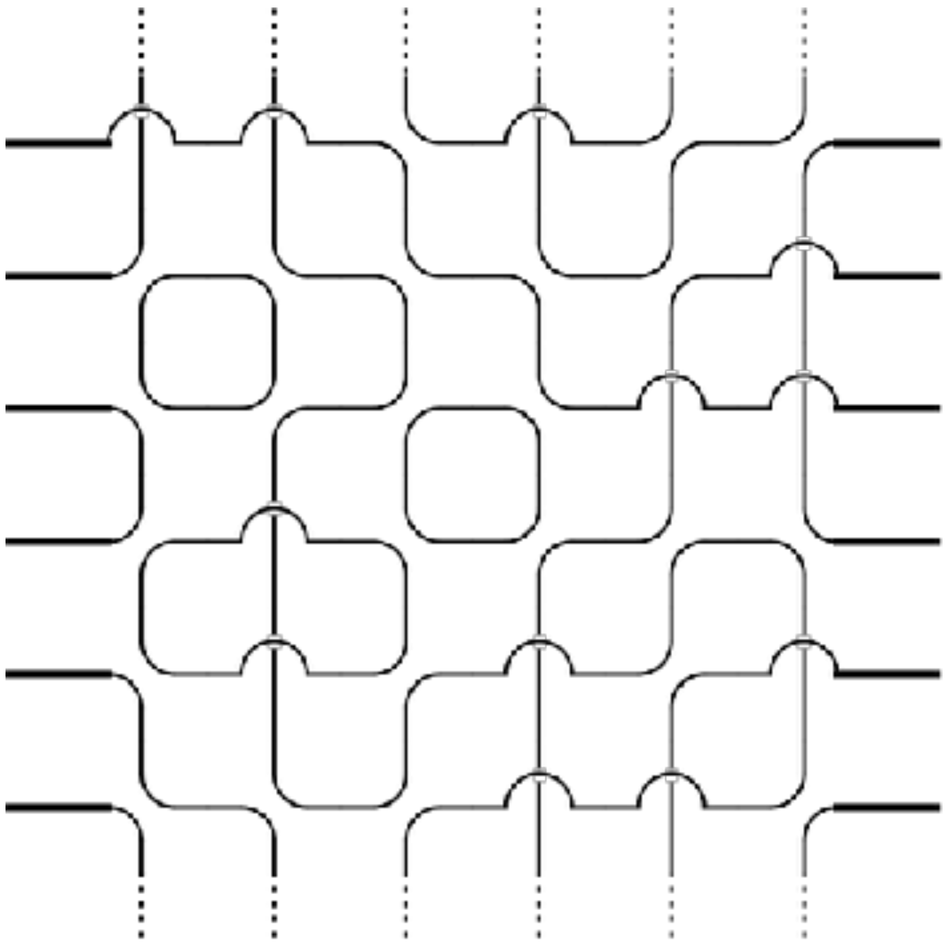


with probability 1/3

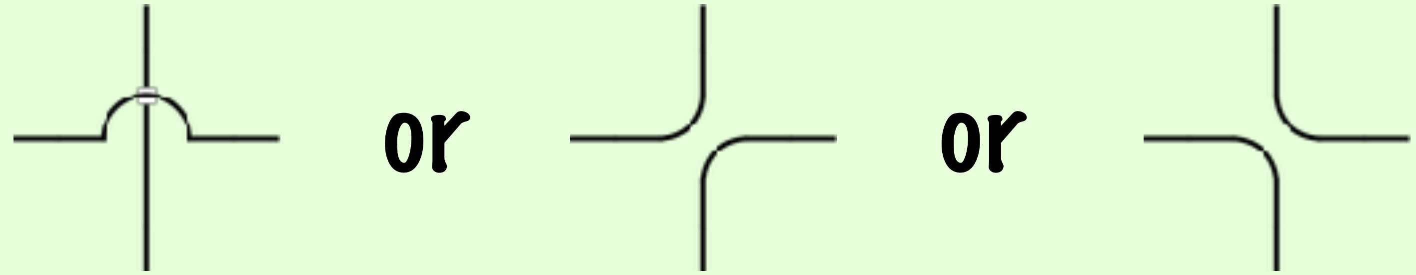


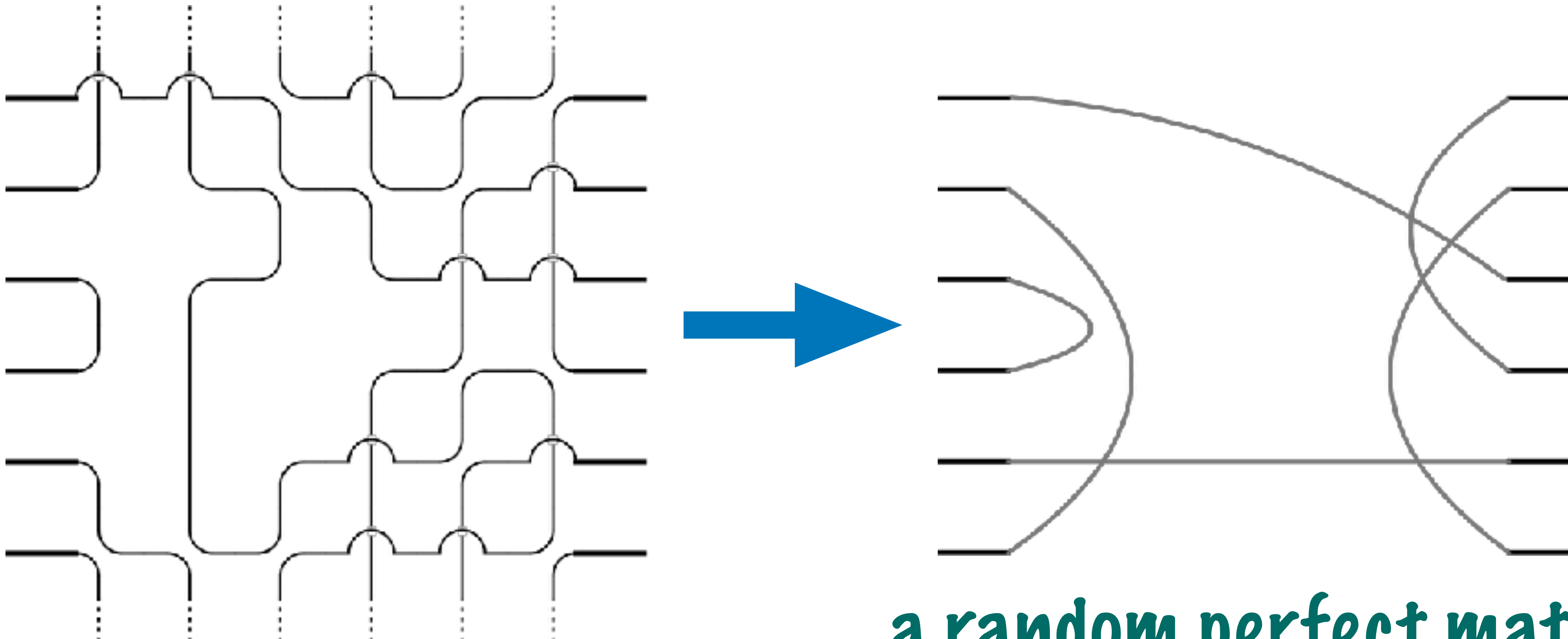
matching of external edges, and crossings

with probability 1/3  or  or 



matching of external edges, and crossings

with probability 1/3 



remove loops

a random perfect matching (pairing) of external edges

left-right pair = crossing

conductance

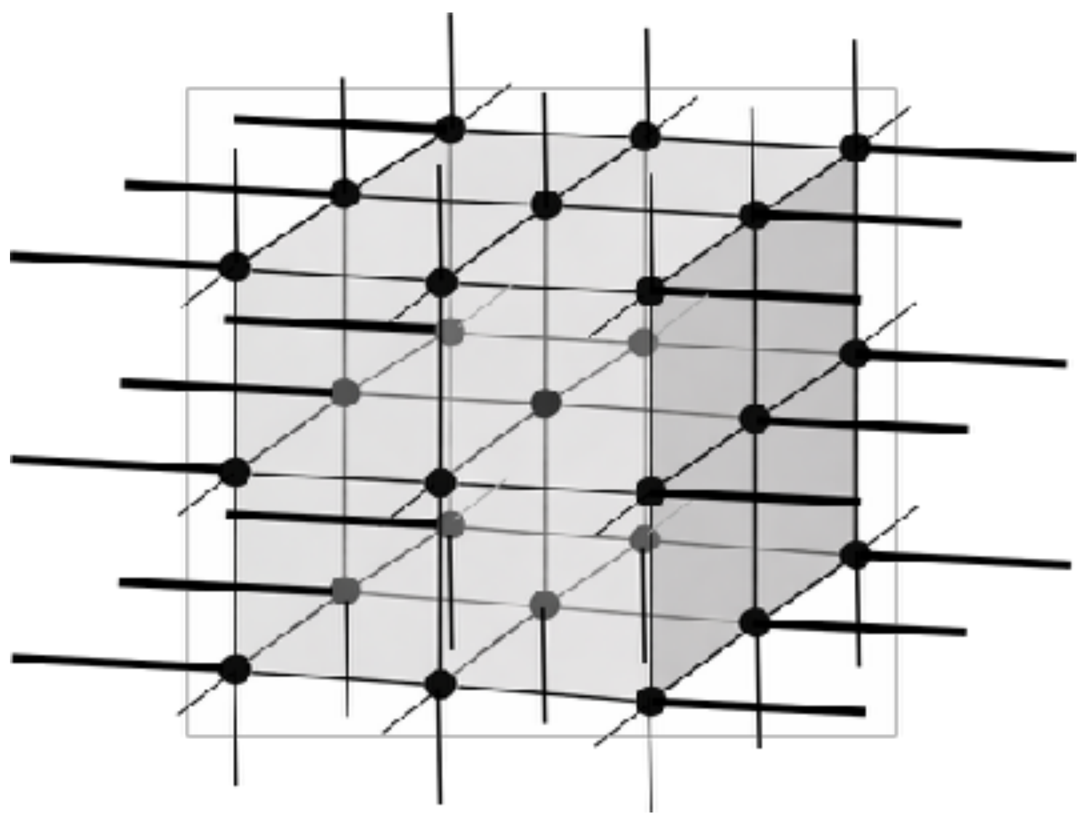
$\mathscr{C}$ the number of crossings

Lorentz mirror model in $d \geq 2$

d dimensional $L \times \cdots \times L$ hyper cubic lattice

horizontal: open b.c., other directions: periodic b.c

add L^{d-1} external edges on the left and the right



Lorentz mirror model in $d \geq 2$

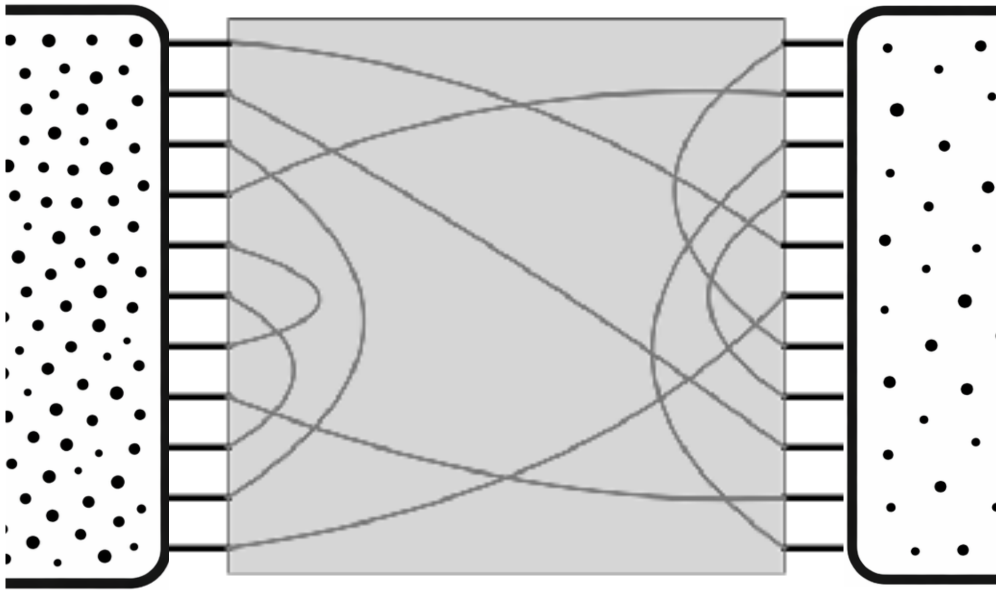
d dimensional $L \times \cdots \times L$ hyper cubic lattice
horizontal: open b.c., other directions: periodic b.c

add L^{d-1} external edges on the left and the right

at each vertex, independently choose one of the $(2d-1)!!$ pairings of the $2d$ incident edges with equal probability

local pairings induce random matching of the $2L^{d-1}$ external edges

conductance $\mathscr{C}$: the number of crossings (matched pairs of left and right external edges)



when left and right boundaries are exposed to particle baths with different densities, the stationary current is proportional to $\mathscr{C}$

Lefevere 2015 , Chiffaudel, Lefevere 2016

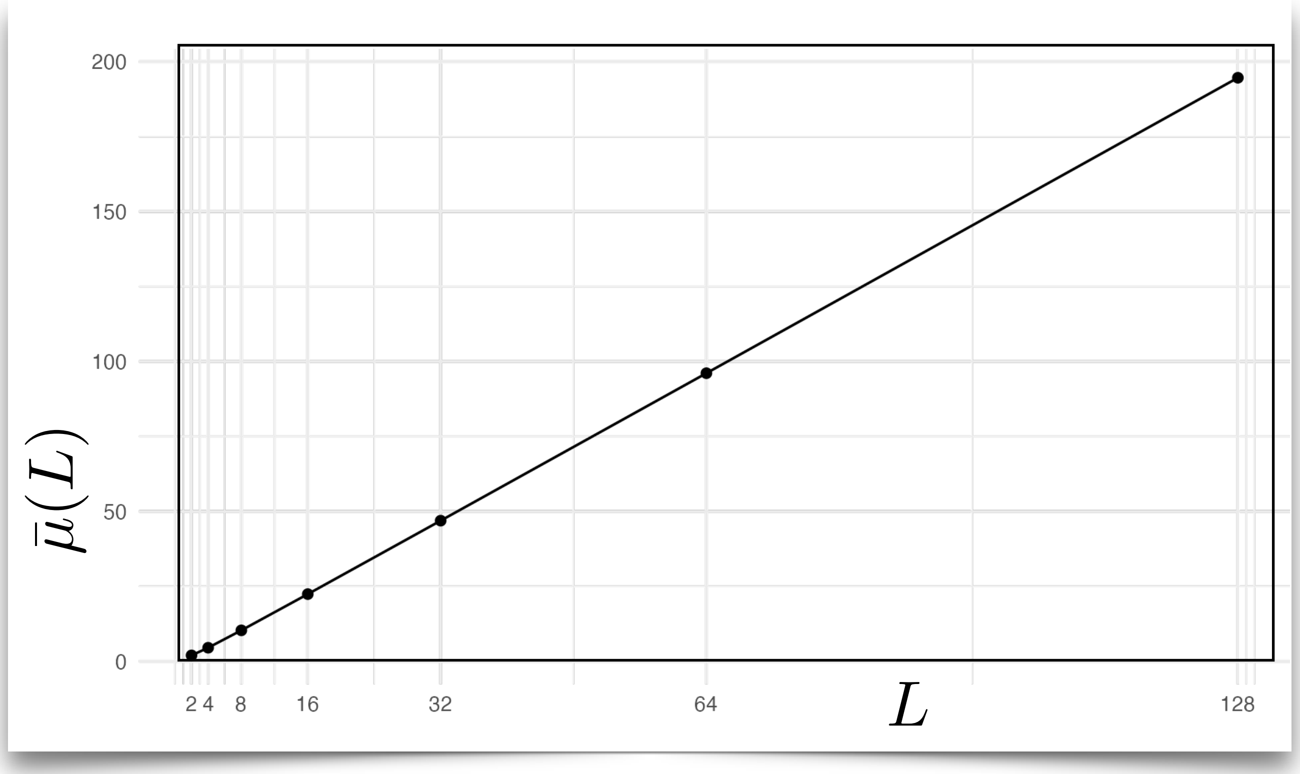
normal transport in the Lorentz mirror model

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

normal transport $$\bar{\mu}(L) \propto \frac{A}{L} = \frac{L^{d-1}}{L} = L^{d-2}$$

numerical and theoretical support for normal transport in $d = 3$

Chiffaudel, Lefevere 2016, Lefevere 2025



normal transport in the Lorentz mirror model

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

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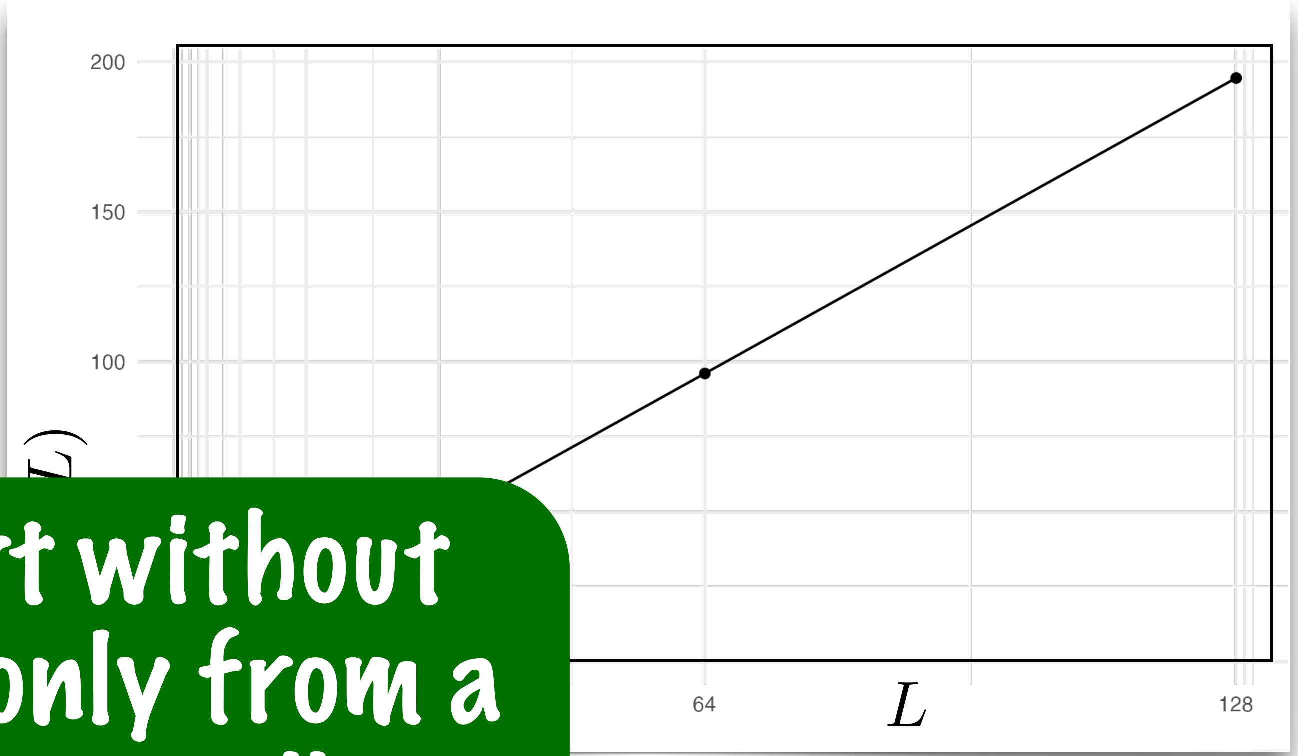


normal (diffusive) transport without stochastic forcing or chaos, only from a quenched random environment!!

the structure of the trajectories is far from diffusive!

rigorous proof seems formidable...

tractable, yet nontrivial, hierarchical version of the model

Lorentz mirror model

Hierarchical model

- definition and recursion relation
- main theorem on normal transport
- Gaussian closure and the "$2/3$ law"

Lorentz mirror model in $d = 3$: numerics

summary

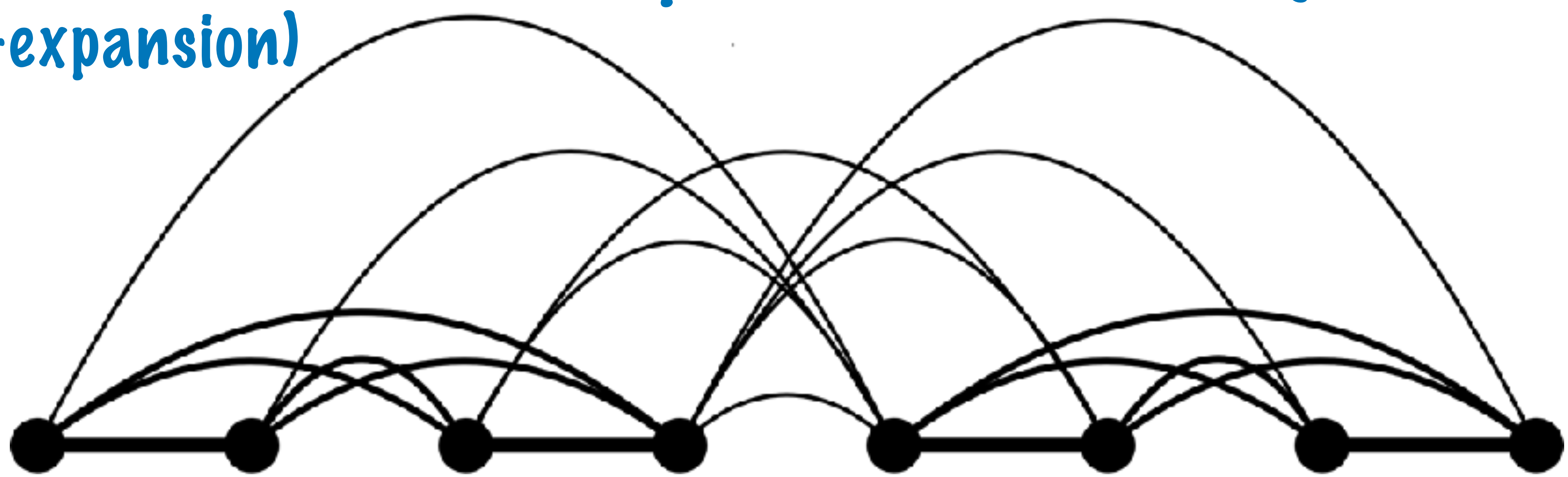
appendix: brief ideas of the proof

hierarchical models in statistical mechanics

Dyson 1969, ...

> Recently Dyson introduced so-called hierarchical models immitating in many respects the lattice systems with pairwise long-range power interaction. (Bleher and Sinai, 1973)

- mean-field-like; interactions are averaged over blocks
  - still preserves the correct dimensional scaling through an explicit hierarchy of length scales
  - predicts the correct upper critical dimension, and also nontrivial critical exponents (to the leading order in the ε-expansion)



Lorentz mirror model

Hierarchical model

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Lorentz mirror model in $d = 3$: numerics

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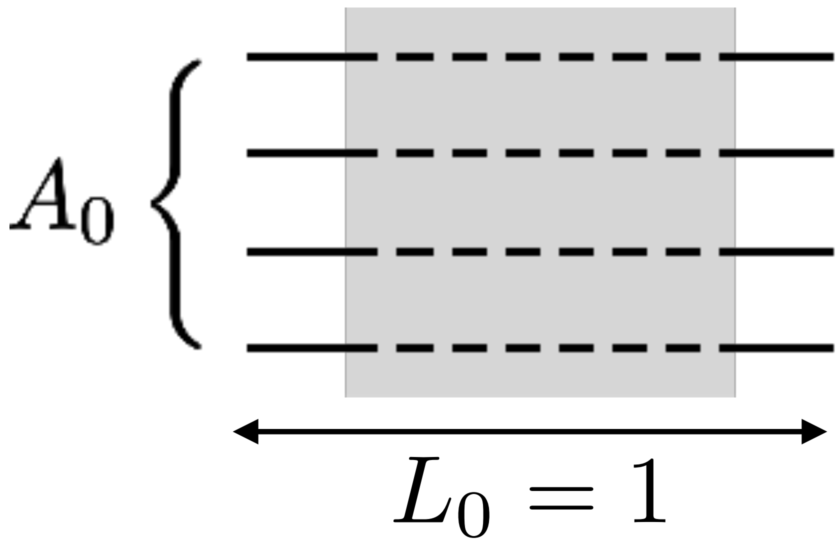
basic setting

dimension $d = 1,2,3,\ldots$ A_0 a positive even integer

generation $n = 0,1,2,\ldots$

generation-0 block



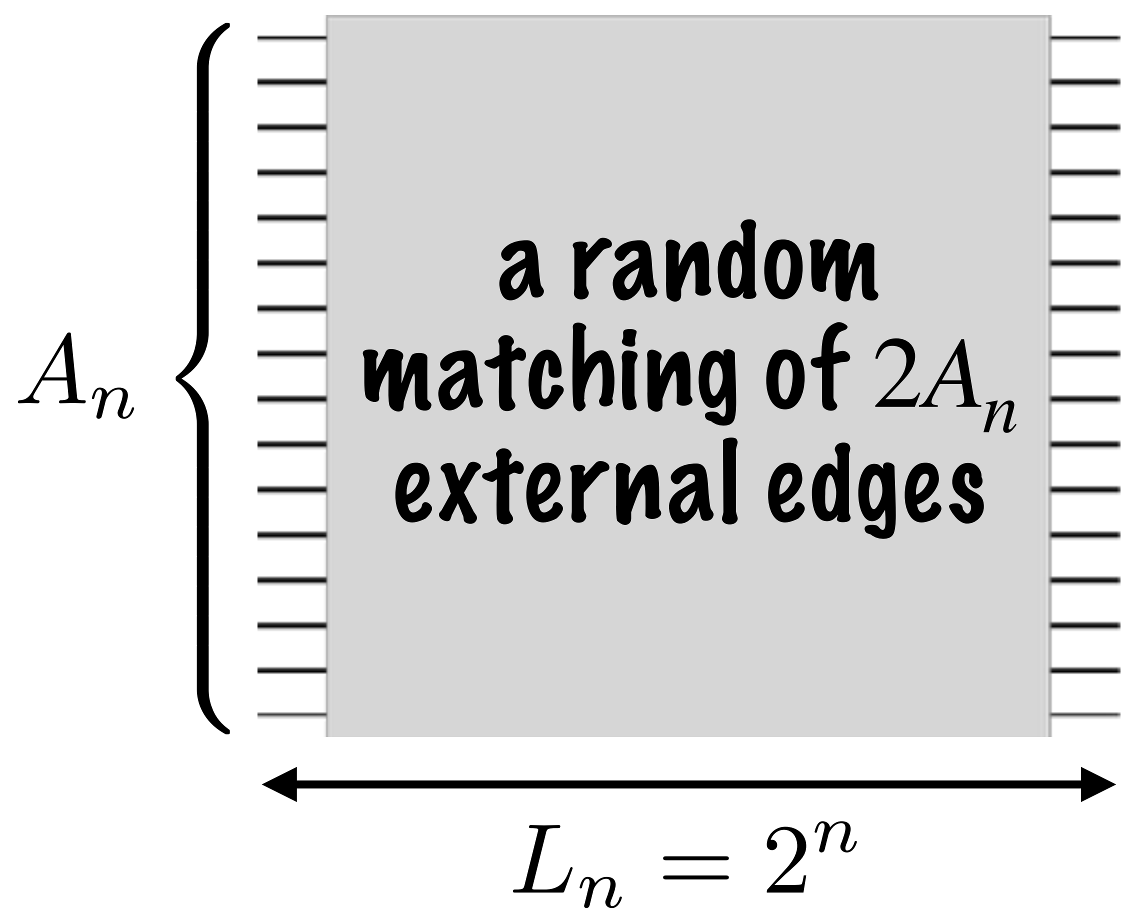
A_0 external edges on the left and right

exactly A_0 crossings

generation-n block

A_n external edges on the left and right

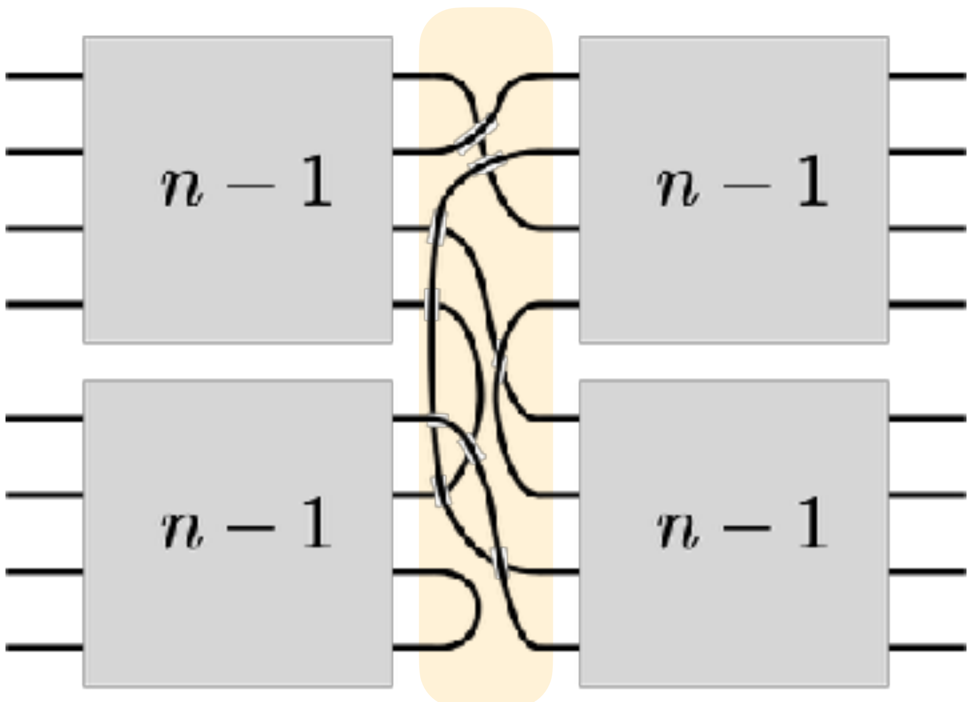
$$A_n = 2^{(d-1)n} A_0$$



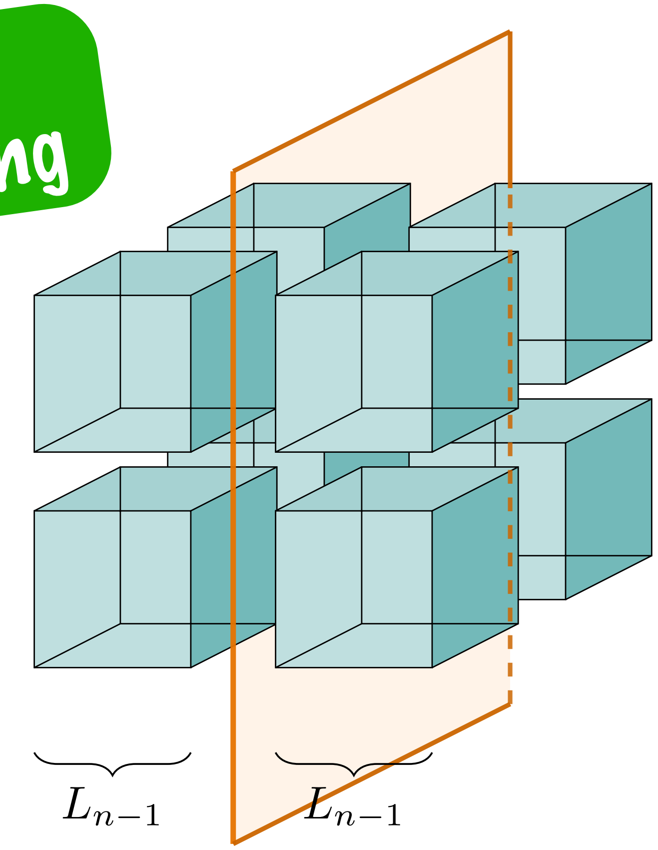
hierarchical construction

2^d independent copies of generation-$(n-1)$ blocks

2^{d-1} copies on the left and right



correct dimensional scaling

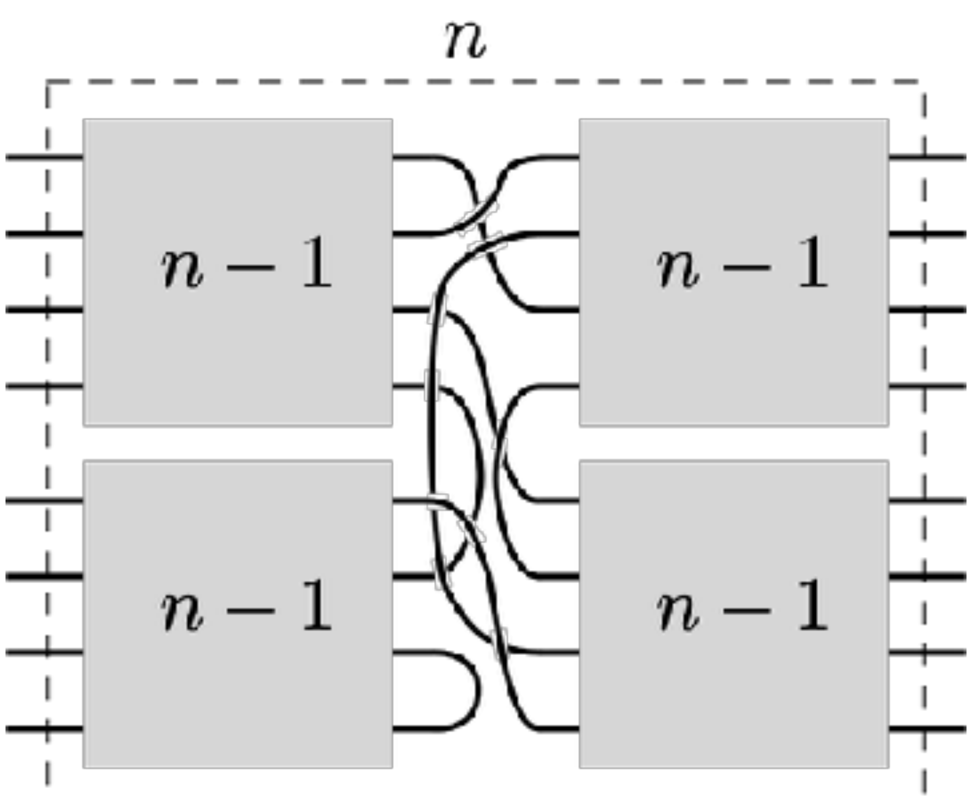


a random matching of $2^d A_{n-1} = 2A_n$ external edges at the interface
(we choose one of the $(2A_n - 1)!!$ perfect matchings with equal probability)

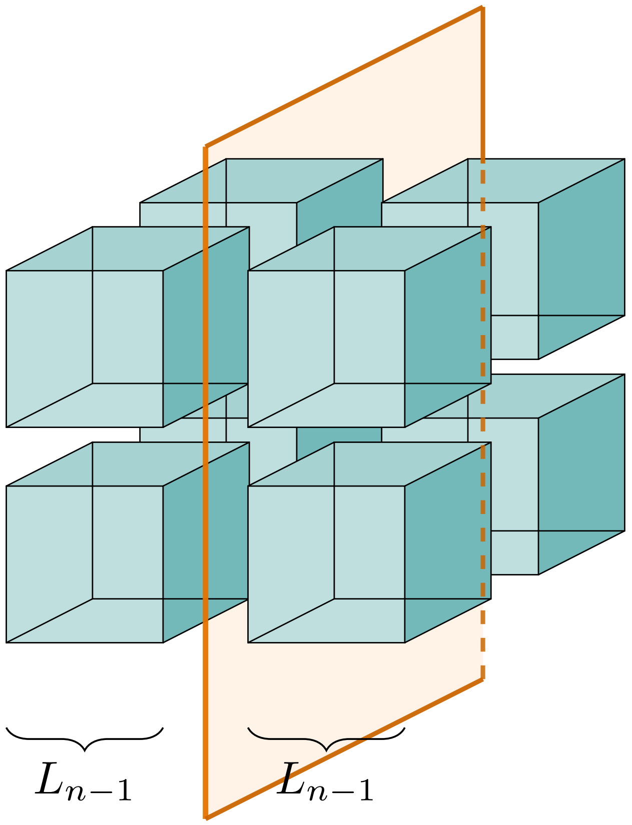
mean-field-like long-range matching

hierarchical construction

2^d independent copies of generation-$(n-1)$ blocks



2^{d-1} copies on the left and right



generation-n block

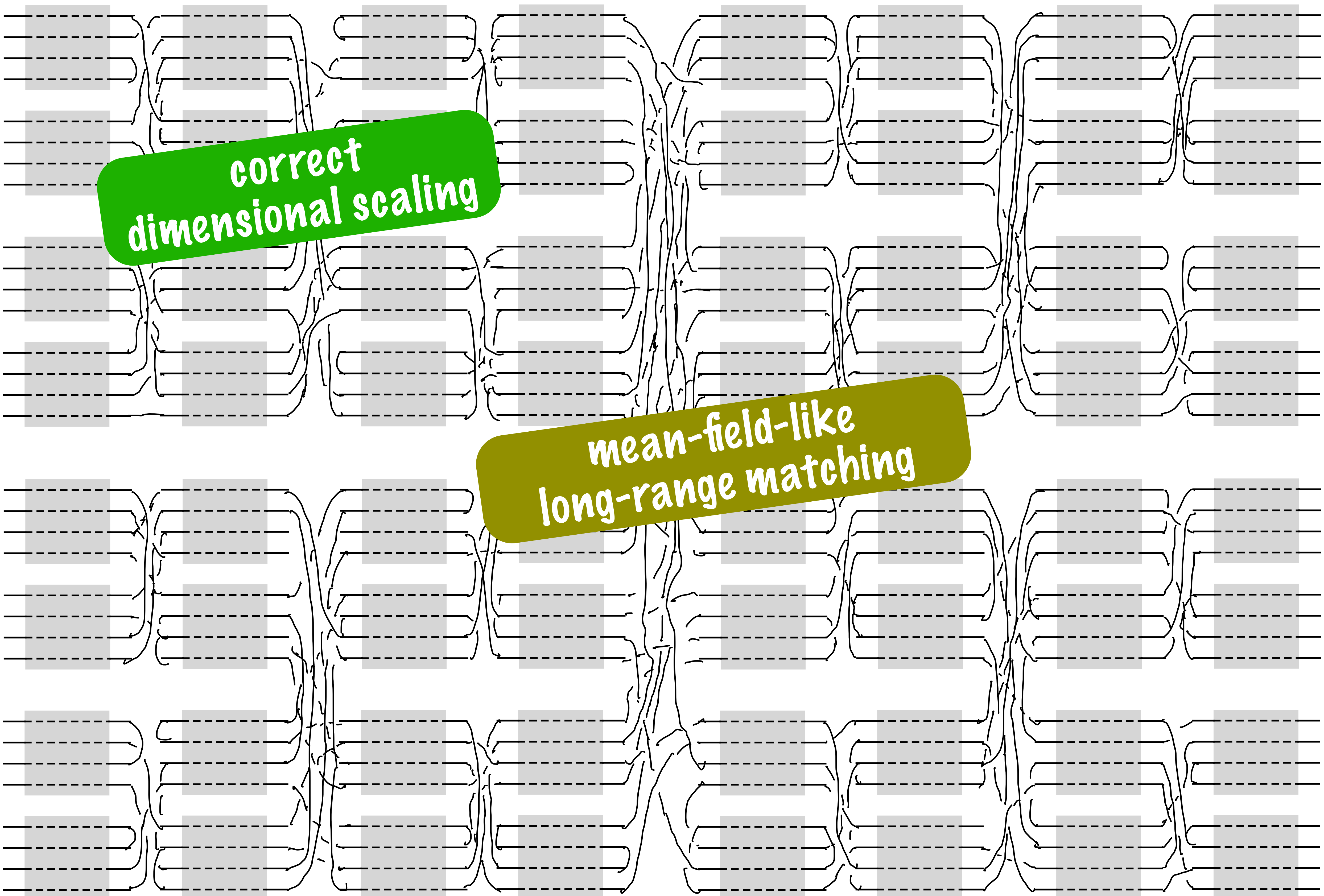
$2^{d-1}A_{n-1} = A_n$ external edges
on the left and right

horizontal length $2L_{n-1} = L_n$

$$A_n = 2^{(d-1)n}A_0 \qquad L_n = 2^n$$

hierarchical construction

a generation-3 block ($d = 2, A_0 = 4$)



exact recursion relation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

$$P_0(\ell) = \delta_{\ell, A_0}$$

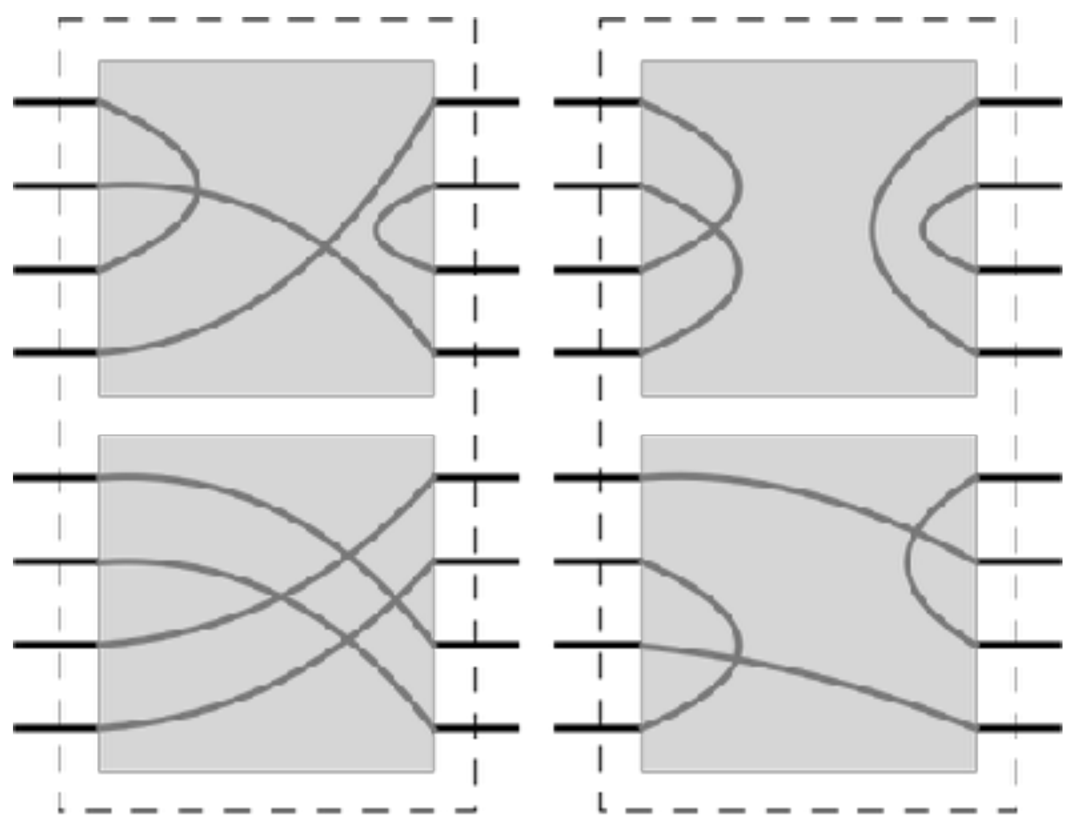
$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2d}} K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

over nonnegative even $\ell_1, \ldots, \ell_{2d}$

$$\ell_{\mathrm{L}} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_{\mathrm{R}} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

the number of crossings on the left

the number of crossings on the right



$\ell_{\mathrm{L}} = 6$

$\ell_{\mathrm{R}} = 2$

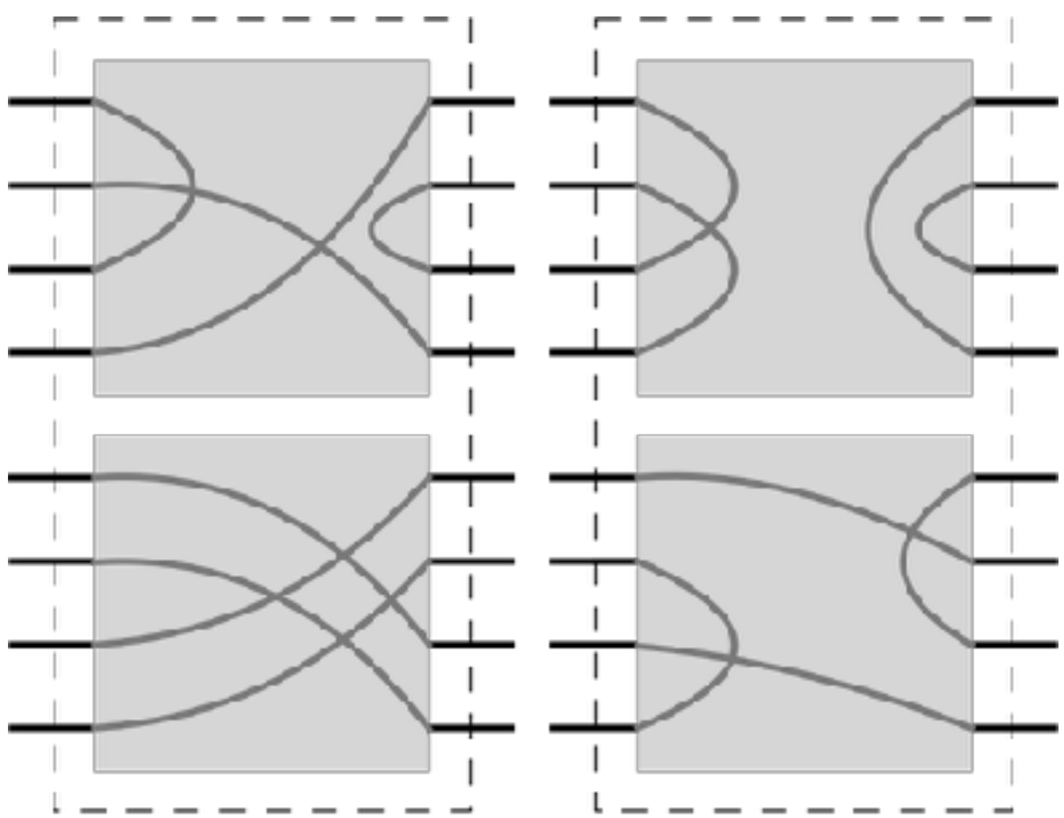
exact recursion relation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

$$P_0(\ell) = \delta_{\ell, A_0}$$

$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2d}} K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

ℓ_{L} crossings on the left



ℓ_{R} crossings on the right

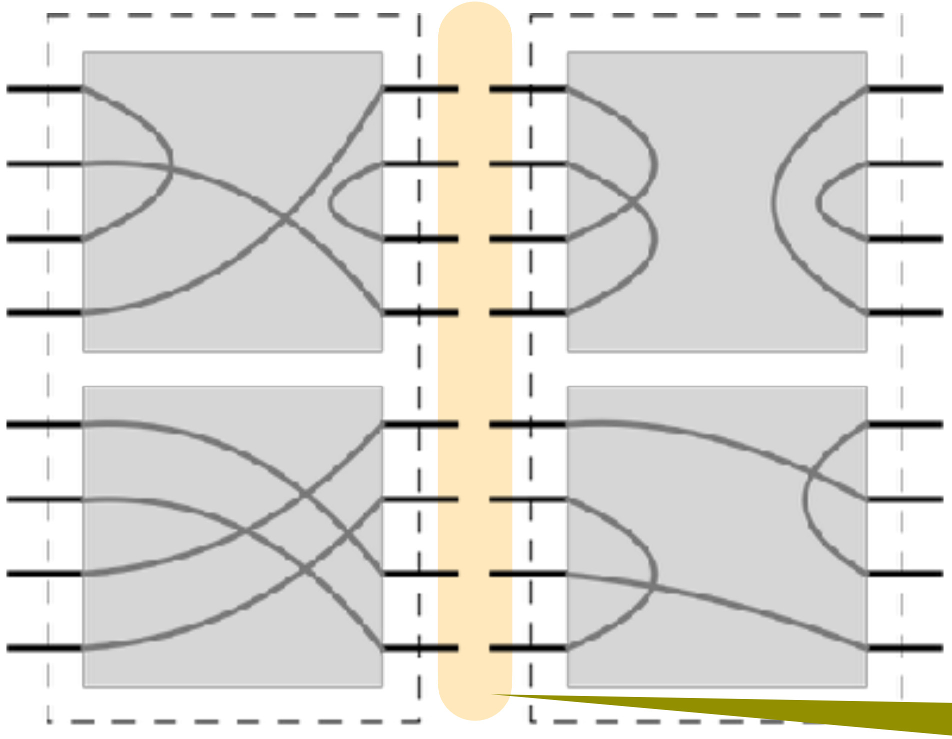
$K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}})$ conditional probability that there are exactly ℓ crossings between the left-most and right-most sides

recursion kernel

$K(\ell \mid \ell_\mathrm{L}, \ell_\mathrm{R})$ conditional probability that there are exactly ℓ crossings between the left-most and right-most sides

ℓ_L crossings on the left

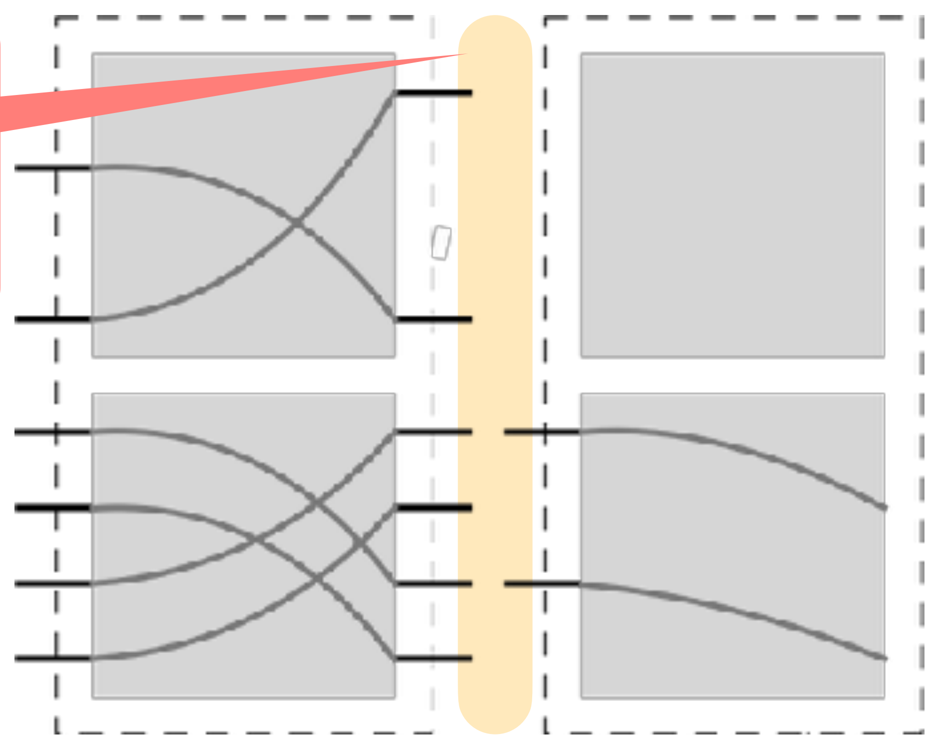
ℓ_R crossings on the right



random perfect matching of all $2A_n$ edges

it suffices to consider

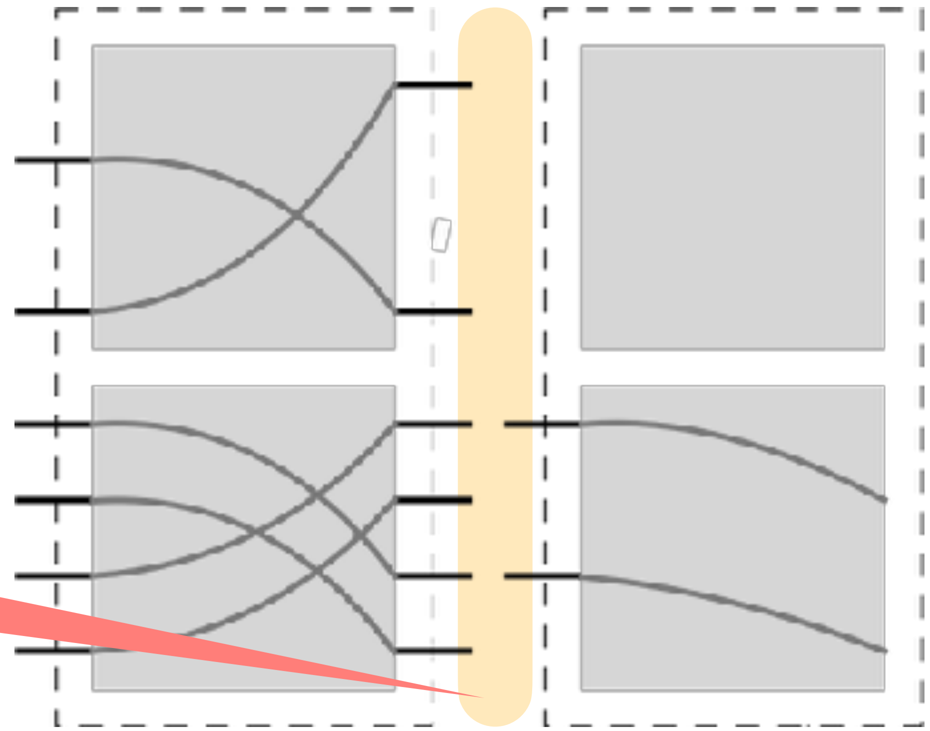
random perfect matching of only $\ell_\mathrm{L} + \ell_\mathrm{R}$ edges



recursion kernel

$K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}})$ conditional probability that there are exactly ℓ crossings between the left-most and right-most sides

ℓ_{L} crossings on the left



ℓ_{R} crossings on the right

random perfect matching of only $\ell_{\mathrm{L}} + \ell_{\mathrm{R}}$ edges

ℓ even integer

$0 \leq \ell \leq \min\{\ell_{\mathrm{R}}, \ell_{\mathrm{R}}\}$

the number of ways to form U-turns

the number of ways to choose ℓ edges from the left and right

$$K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = \frac{(\ell_{\mathrm{L}} - \ell - 1)!!(\ell_{\mathrm{R}} - \ell - 1)!!}{(\ell_{\mathrm{L}} + \ell_{\mathrm{R}} - 1)!!} \binom{\ell_{\mathrm{L}}}{\ell} \binom{\ell_{\mathrm{R}}}{\ell} \ell!$$

the total number of matchings of $\ell_{\mathrm{L}} + \ell_{\mathrm{R}}$ edges

the number of ways to match those edges

exact recursion relation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

$$P_0(\ell) = \delta_{\ell, A_0}$$

$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2d}} K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

over nonnegative even $\ell_1, \ldots, \ell_{2d}$

$$\ell_{\mathrm{L}} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_{\mathrm{R}} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

$$K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = \frac{(\ell_{\mathrm{L}} - \ell - 1)!!(\ell_{\mathrm{R}} - \ell - 1)!!}{(\ell_{\mathrm{L}} + \ell_{\mathrm{R}} - 1)!!} \binom{\ell_{\mathrm{L}}}{\ell} \binom{\ell_{\mathrm{R}}}{\ell} \ell!$$

if ℓ is an even integer s.t. $0 \le \ell \le \min\{\ell_{\mathrm{R}}, \ell_{\mathrm{R}}\}$

$K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = 0$ otherwise

$$\sum_\ell K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = 1$$

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behavior of the mean conductance

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

mean conductance $\mu_n = \sum_{\ell=0,2,\ldots} \ell\, P_n(\ell)$

conditional mean

for $\ell_\mathrm{L}, \ell_\mathrm{R} \gg 1$

$$\tilde{\mu}(\ell_\mathrm{L}, \ell_\mathrm{R}) = \sum_{\ell} \ell\, K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) = \frac{\ell_\mathrm{L}\, \ell_\mathrm{R}}{\ell_\mathrm{L} + \ell_\mathrm{R} + 1} \simeq \Big(\frac{1}{\ell_\mathrm{L}} + \frac{1}{\ell_\mathrm{R}}\Big)^{-1}$$

since the main contribution in the recursion should come from $\ell_\mathrm{L} \simeq \ell_\mathrm{R} \simeq 2^{d-1}\mu_{n-1}$, we expect

$$\mu_n \simeq 2^{d-2}\mu_{n-1}$$

$$\mu_n \simeq 2^{(d-2)n}\mu_0 = 2^{(d-2)n}A_0 = \frac{A_n}{L_n}$$

normal transport!!

$$A_n = 2^{(d-1)n}A_0$$
$$L_n = 2^n$$

$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$\ell_\mathrm{L} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_\mathrm{R} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

main theorem: normal transport

theorem: let $d \geq 3$. for sufficiently large A_0 there exist positive constants C_d, C'_d such that

$$C_d \frac{A_n}{L_n} - 1 \leq \mu_n \leq C'_d \frac{A_n}{L_n}$$

for all n. if $A_0 \gg 1$, one has $C_d \simeq C'_d \simeq 1$ and hence $\mu_n \simeq A_n/L_n$ for all n.

for $d = 3$, $A_0 \geq 2$ suffices (no conditions)
$C_3 = 1 - 5/(4A_0)$, $C'_3 = 1 + 1/(3A_0)$

normal transport from non-chaotic deterministic motion in a quenched random environment

proposition: let $d = 1$ and assume $A_0 \gg 1$. then $\mu_n \simeq A_0/L_n$ as long as μ_n is larger than $O(1)$

Lorentz mirror model

Hierarchical model

- definition and recursion relation
- main theorem on normal transport
- Gaussian closure and the "$2/3$ law"

Lorentz mirror model in $d = 3$: numerics

summary

appendix: brief ideas of the proof

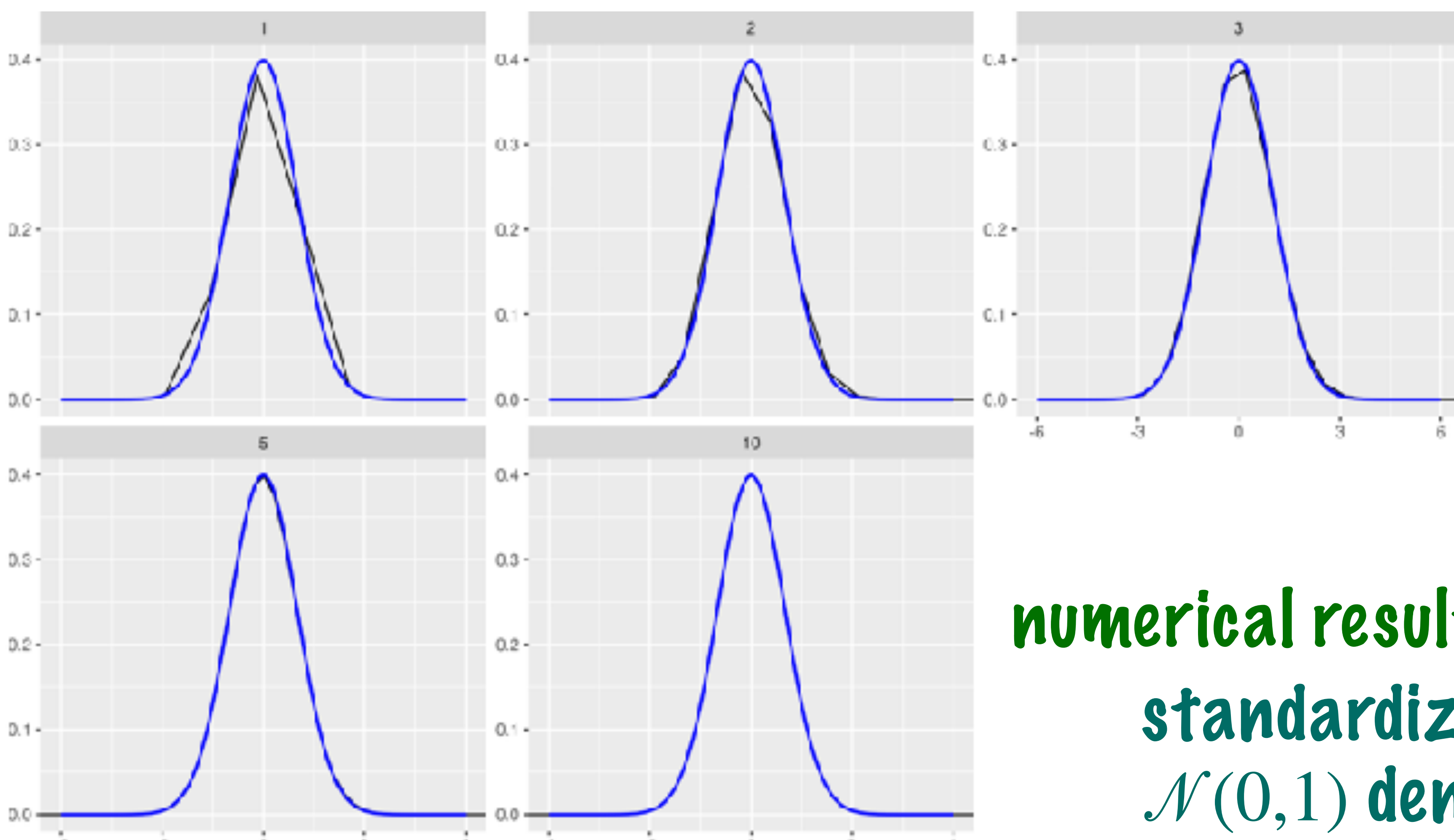
basic ansatz and approximation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

mean conductance $\mu_n = \sum_{\ell=0,2,\ldots} \ell \, P_n(\ell)$

sample-to-sample variance $v_n = \sum_{\ell=0,2,\ldots} (\ell - \mu_n)^2 P_n(\ell)$

ansatz $P_n(\ell) \propto \exp\Big[-\dfrac{(\ell-\mu_n)^2}{2v_n}\Big]$



numerical results for $d = 3$ with $A_0 = 2$

standardized $P_n(\ell)$ vs $\mathcal{N}(0,1)$ density for $n = 1,2,3,5,10$

basic ansatz and approximation

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approximation

$$K(\ell\,|\,\ell_{\rm L},\ell_{\rm R}) = \frac{(\ell_{\rm L}-\ell-1)!!(\ell_{\rm R}-\ell-1)!!}{(\ell_{\rm L}+\ell_{\rm R}-1)!!}\binom{\ell_{\rm L}}{\ell}\binom{\ell_{\rm R}}{\ell}\ell!$$

$$\propto \exp\Big[-\frac{\{\ell-\tilde{\mu}(\ell_{\rm L},\ell_{\rm R})\}^2}{2\,\tilde{v}(\ell_{\rm L},\ell_{\rm R})}\Big]$$

$$\tilde{\mu}(\ell_{\rm L},\ell_{\rm R}) = \frac{\ell_{\rm L}\,\ell_{\rm R}}{\ell_{\rm L}+\ell_{\rm R}+1} \qquad \tilde{v}(\ell_{\rm L},\ell_{\rm R}) = \frac{2\ell_{\rm L}\ell_{\rm R}(\ell_{\rm L}-1)(\ell_{\rm R}-1)}{(\ell_{\rm L}+\ell_{\rm R}-3)(\ell_{\rm L}+\ell_{\rm R}-1)^2}$$

recursion relations for μ_n and v_n

ansatz $$P_n(\ell) \propto \exp\Bigl[-\frac{(\ell-\mu_n)^2}{2v_n}\Bigr]$$

approximation $$K(\ell\,|\,\ell_{\rm L},\ell_{\rm R}) \propto \exp\Bigl[-\frac{\{\ell-\tilde{\mu}(\ell_{\rm L},\ell_{\rm R})\}^2}{2\,\tilde{v}(\ell_{\rm L},\ell_{\rm R})}\Bigr]$$

$$\tilde{\mu}(\ell_{\rm L},\ell_{\rm R}) = \frac{\ell_{\rm L}\,\ell_{\rm R}}{\ell_{\rm L}+\ell_{\rm R}+1} \qquad \tilde{v}(\ell_{\rm L},\ell_{\rm R}) = \frac{2\ell_{\rm L}\ell_{\rm R}(\ell_{\rm L}-1)(\ell_{\rm R}-1)}{(\ell_{\rm L}+\ell_{\rm R}-3)(\ell_{\rm L}+\ell_{\rm R}-1)^2}$$

recursion $$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell\,|\,\ell_{\rm L},\ell_{\rm R})\,\prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$\ell_{\rm L} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_{\rm R} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

for $d \geq 3$, the leading terms are

$$\mu_n \simeq 2^{d-2}\mu_{n-1}$$

$$v_n \simeq 2^{d-3}\mu_{n-1} + 2^{d-4}v_{n-1}$$

normal transport

for $d \geq 3$, the leading terms are

$$\mu_n \simeq 2^{d-2}\mu_{n-1} \qquad v_n \simeq 2^{d-3}\mu_{n-1} + 2^{d-4}v_{n-1}$$

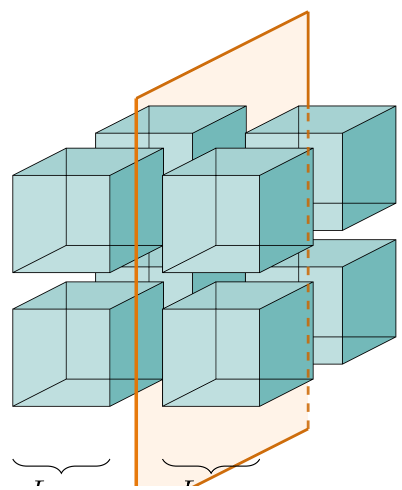
$$\mu_n \simeq (\text{const})2^{(d-2)n} = \kappa\frac{A_n}{L_n} \text{ as } n \uparrow \infty$$

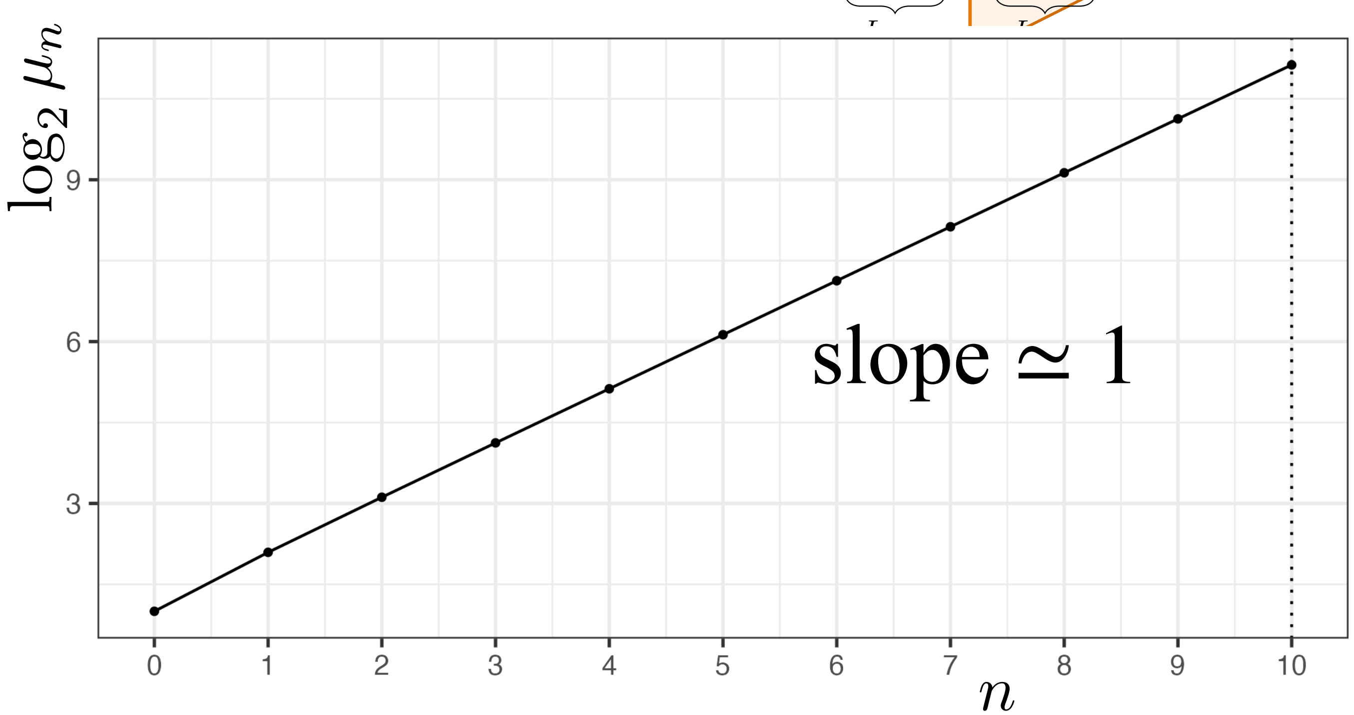
conductivity

normal transport

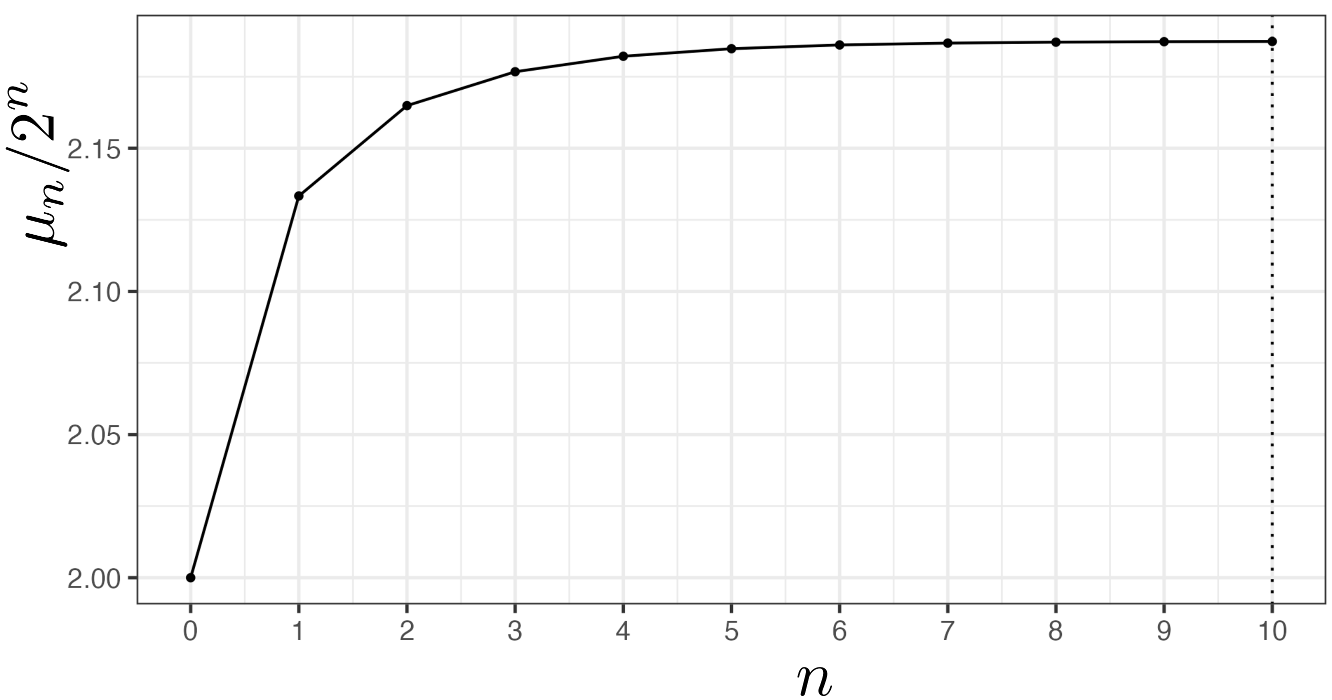
proven rigorously!

numerical results for $d = 3$ with $A_0 = 2$





$\kappa \simeq 1.1$



universal "2/3 law"

for $d \geq 3$, the leading terms are

fluctuation from random matching

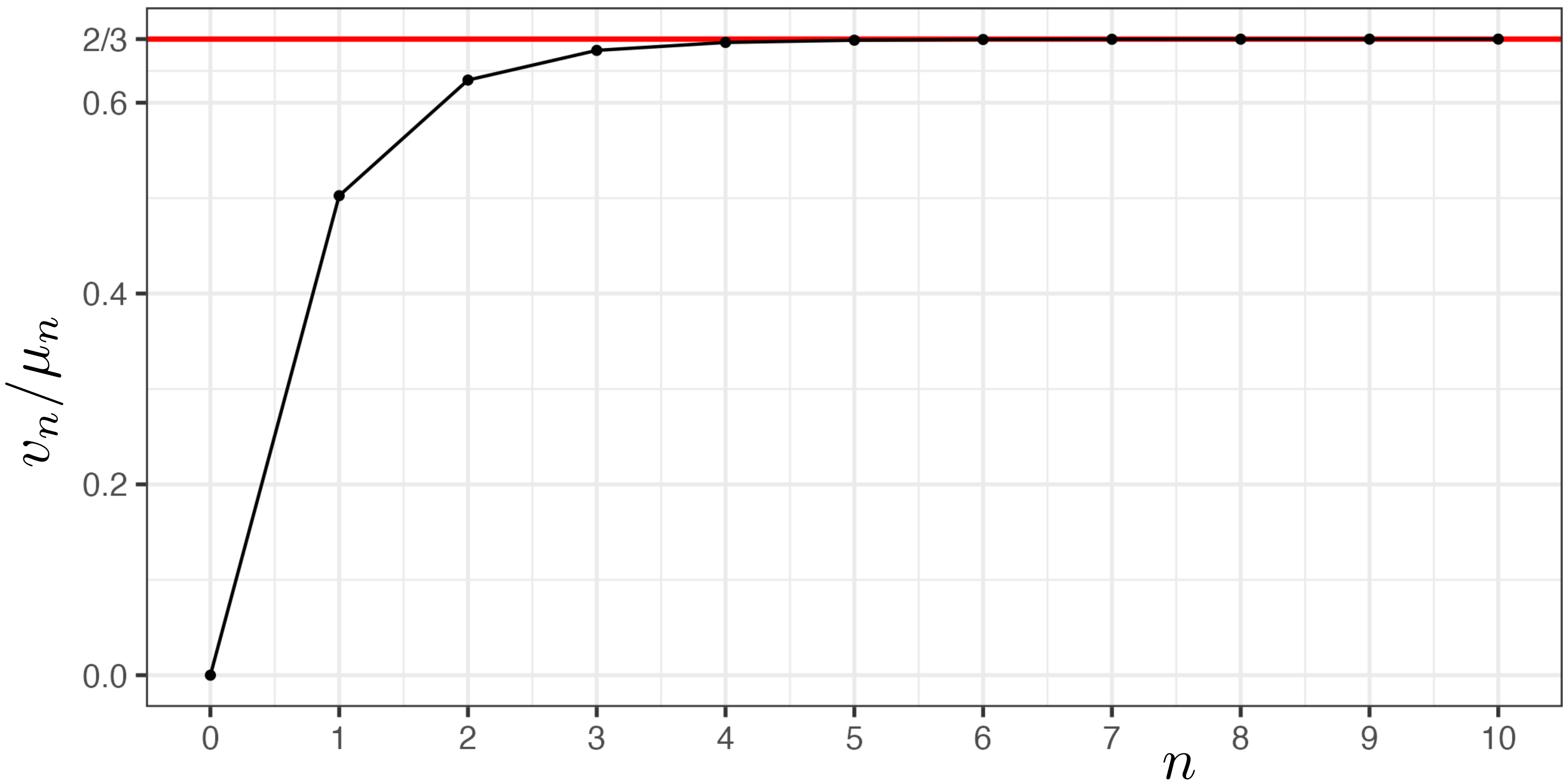
fluctuation from generation $n-1$

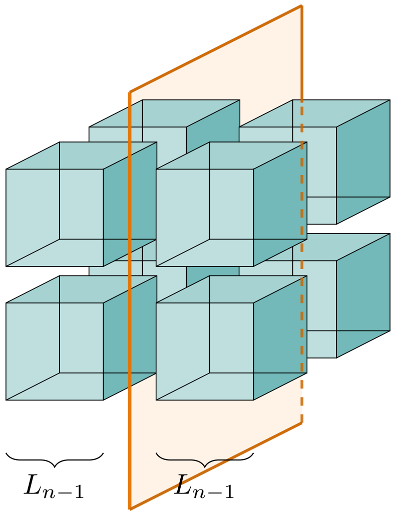
$$\mu_n \simeq 2^{d-2}\mu_{n-1} \qquad v_n \simeq 2^{d-3}\mu_{n-1} + 2^{d-4}v_{n-1}$$

$$\frac{v_n}{\mu_n} \simeq \frac{1}{2} + \frac{1}{4}\frac{v_{n-1}}{\mu_{n-1}}$$

d-independent recursion relation

$\frac{v_n}{\mu_n} \to \frac{2}{3}$ rapidly as $n \uparrow \infty$ in any dimension $d \geq 3$





numerical results for $d = 3$ with $A_0 = 2$

marginal dimension $d = 2$

for $d = 2$, we include higher order contributions

$$\mu_n \simeq \mu_{n-1} + \frac{1}{4} - \frac{v_{n-1}}{4\mu_{n-1}} \qquad v_n \simeq \frac{\mu_{n-1}}{2} + \frac{v_{n-1}}{4}$$

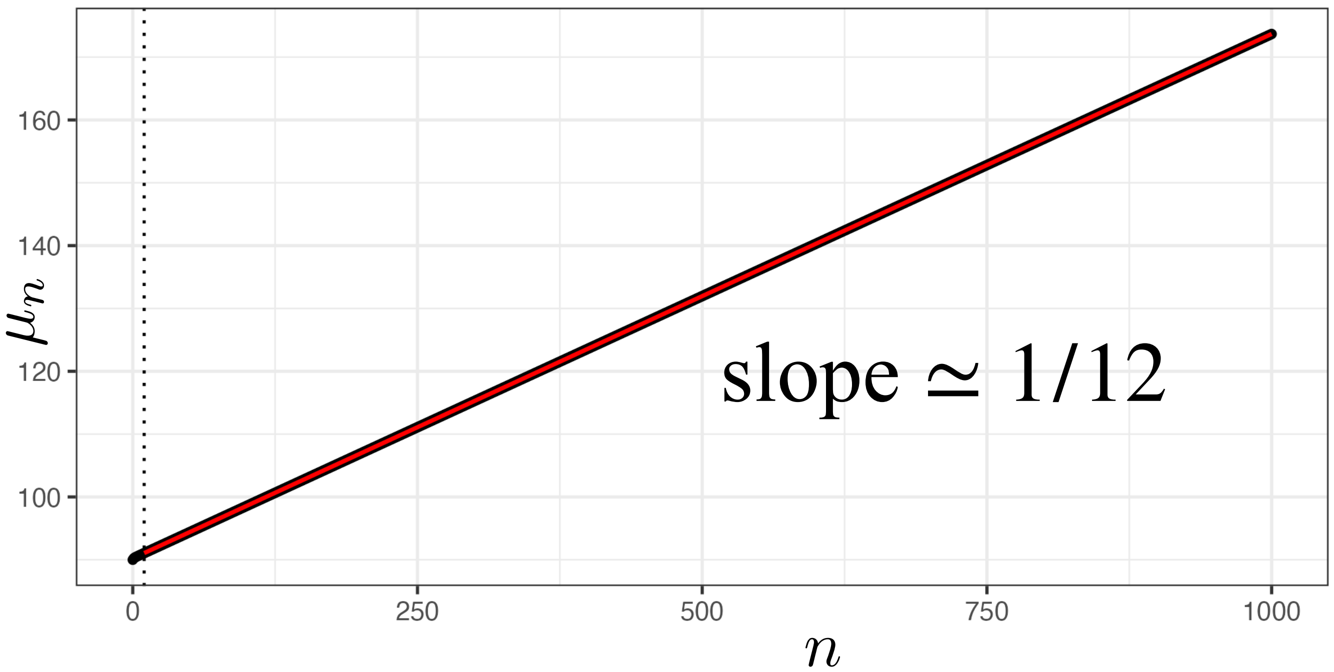
as $n \uparrow \infty$ $\quad \mu_n \simeq \frac{n}{12} = \frac{\log L_n}{12\,\log 2}$

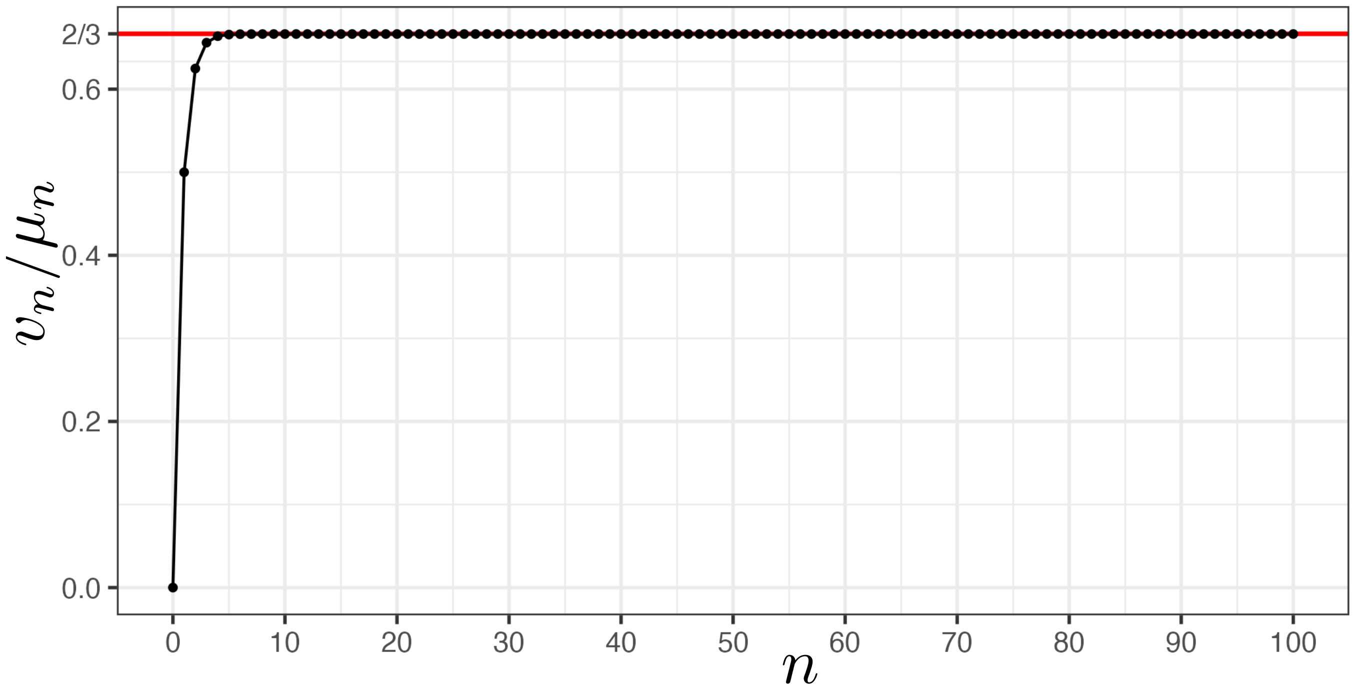
weakly anomalous transport with a logarithmic correction

$v_n \simeq \frac{n}{18}$ and hence $\frac{v_n}{\mu_n} \to \frac{2}{3}$

2/3 law!!

numerical results for $d = 2$ with $A_0 = 90$





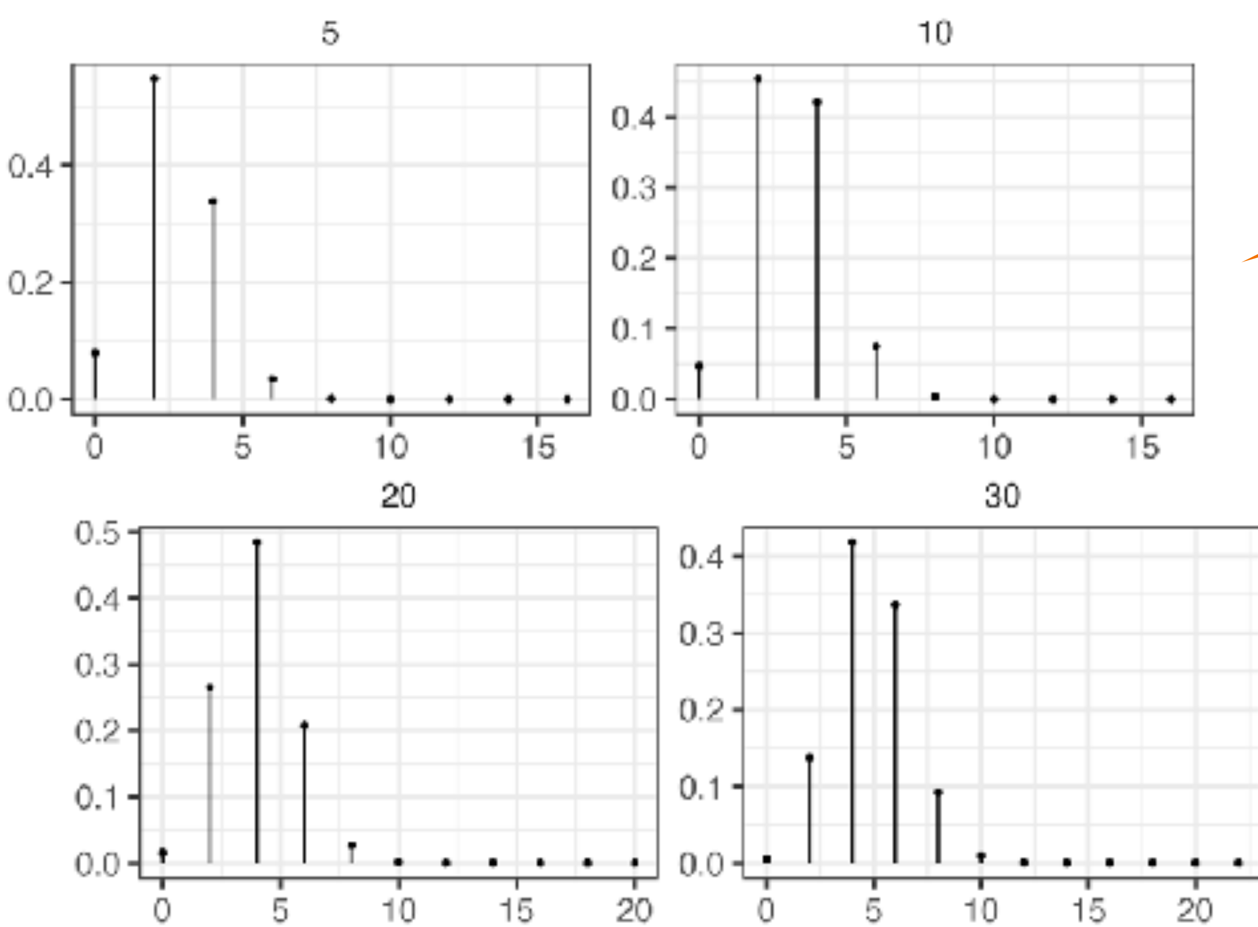
robustness of the "2/3 law"

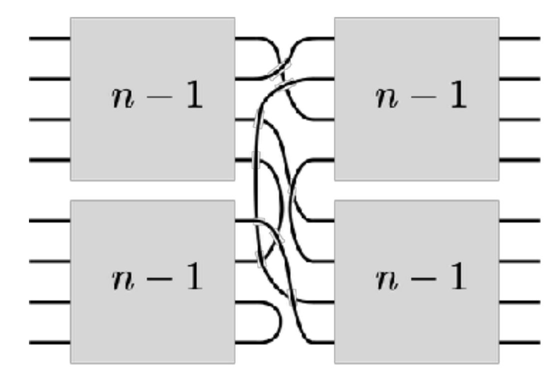
$$\frac{v_n}{\mu_n} \to \frac{2}{3} \text{ rapidly whenever } \mu_n \uparrow \infty$$

"2/3 law" may be much more robust than the Gaussianity

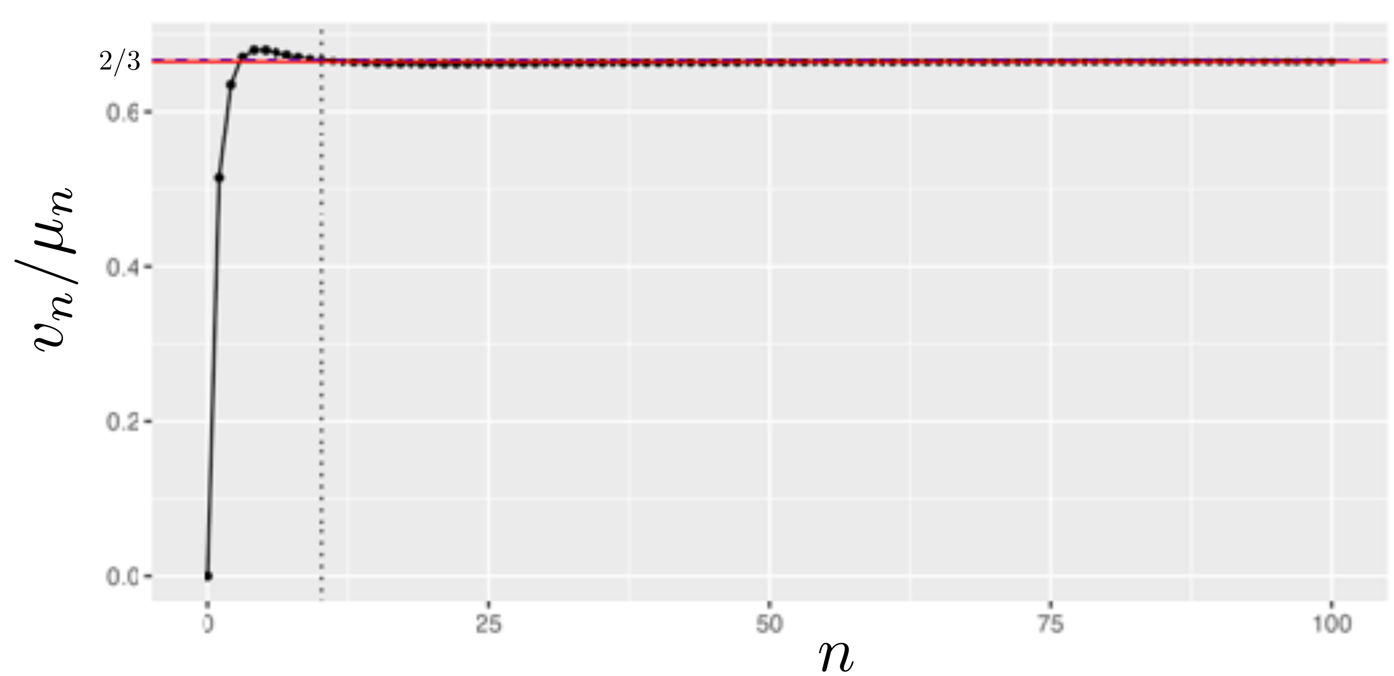
numerical results for $d = 2$ with $A_0 = 2$

$P_n(\ell)$ vs ℓ for $n = 5,10,20,30$





very far from Gauss!!!



Lorentz mirror model

Hierarchical model

- definition and recursion relation
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Lorentz mirror model in $d = 3$: numerics

summary

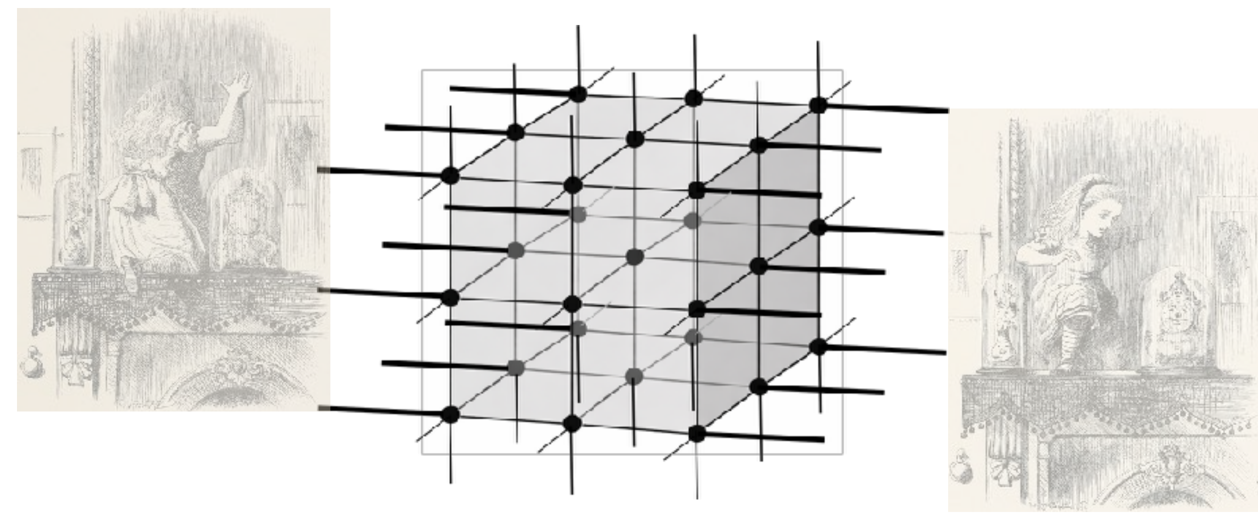
appendix: brief ideas of the proof

Monte Carlo simulation of the original Lorentz mirror model on the cubic lattice

the $L \times L \times L$ cubic lattice

$L = 2^n$ with $n = 1,2,\ldots,7$

choose one of the $5!! = 15$ pairings at each vertex

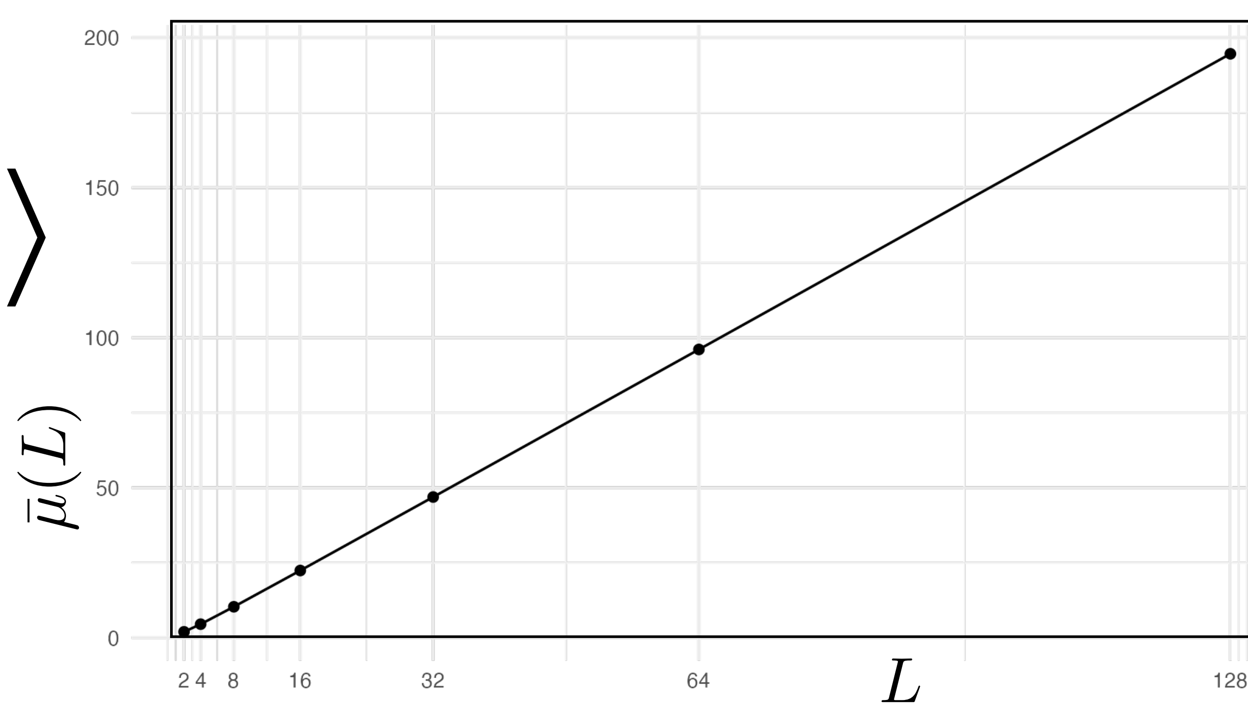


generate 6.4×10^5 independent environments

for each realization of the environment, faithfully count the number of crossings $\mathscr{C}$, i.e., the conductance

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

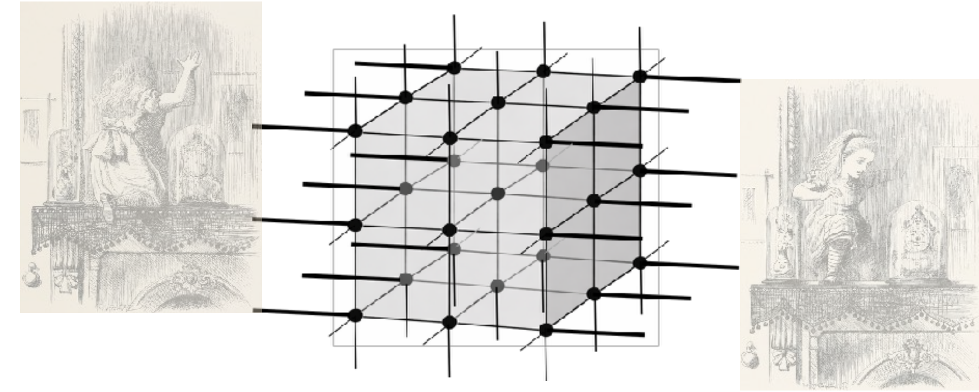
variance $\bar{v}(L) = \left\langle \left(\mathscr{C} - \bar{\mu}(L) \right)^2 \right\rangle$



we recover normal transport

Chiffaudel, Lefevere 2016

variance-to-mean ratio



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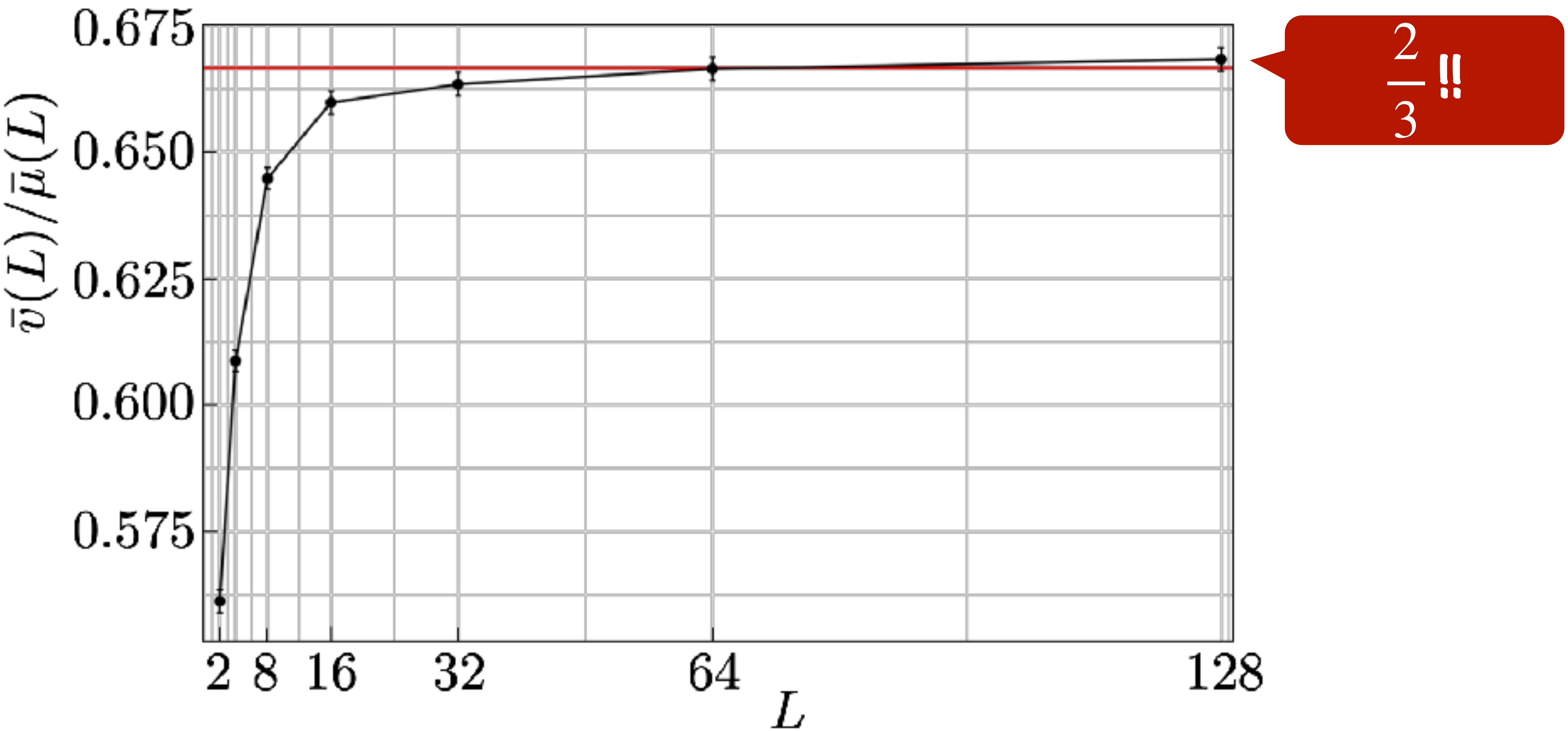
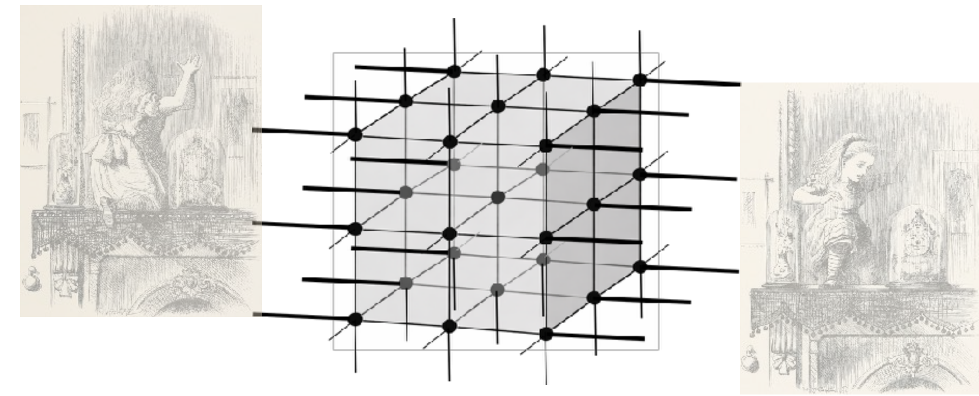


Monte Carlo with 6.4×10^5 samples
error bars indicate 95% confidence intervals

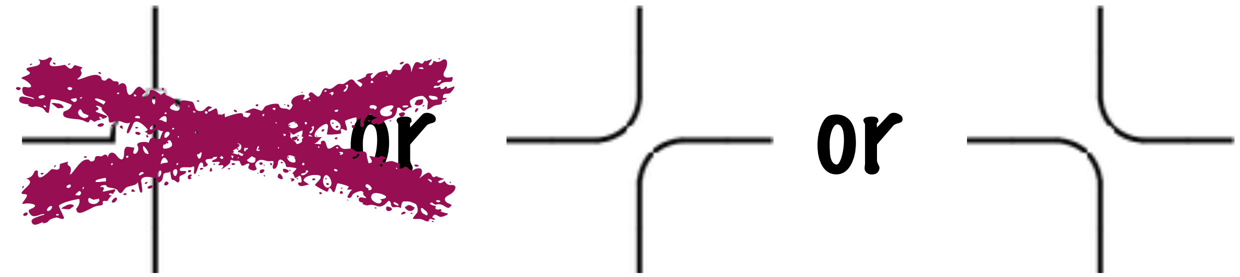
variance-to-mean ratio

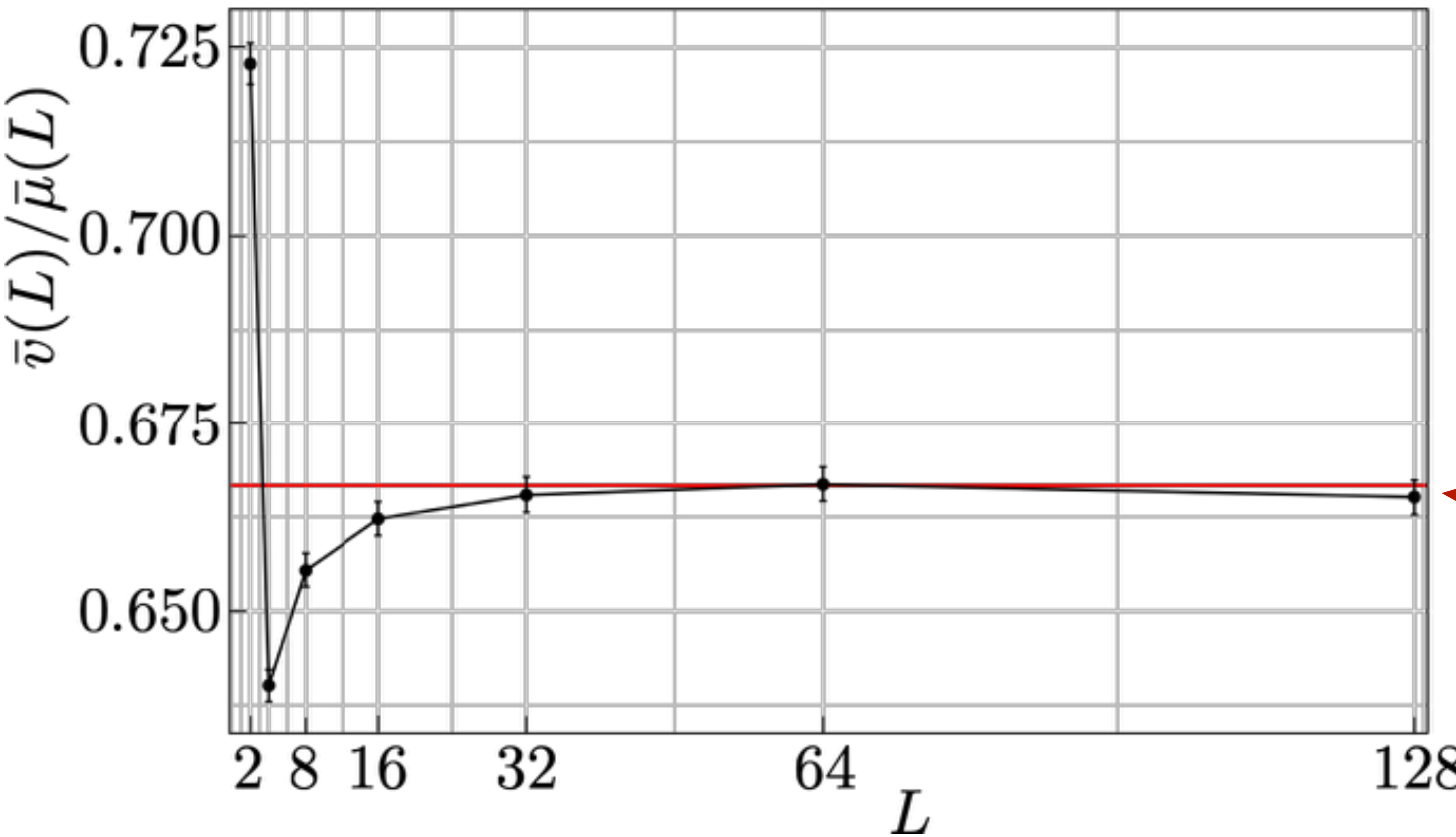


"orthogonal rule"

particles never go straight

choose one of the 8 orthogonal pairings at each vertex





still $\frac{2}{3}$!!!!

Monte Carlo with 6.4×10^5 samples

error bars indicate 95% confidence intervals

the universal "2/3 law"

$$\frac{\text{(sample-to-sample variance of conductance)}}{\text{(mean conductance)}} \to \frac{2}{3}$$

observed theoretically (but not proved!) in the hierarchical Lorentz mirror model in $d \geq 2$, and numerically in the original Lorentz mirror model in $d = 3$

conjecture: ν/μ (as $\mu \uparrow \infty$) is a universal amplitude ratio, and the value $2/3$ signifies a universality class of systems (including ours) exhibiting normal transport induced by random matching of conserved current

what models belong to this universality class?

any models in the continuum as in the original Lorentz gas?

Lorentz mirror model

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Lorentz mirror model in $d=3$: numerics

summary

appendix: brief ideas of the proof

summary

- ☑ the Lorentz mirror model (both original and hierarchical) provides a clean setting to study transport generated by a quenched random environment
  - ☑ in the hierarchical version, we proved normal transport in $d \geq 3$ and showed (but didn't prove) weakly anomalous transport with log-correction in $d = 2$
  - ☑ in the original model, normal transport has been expected for $d \geq 3$. we conjecture weakly anomalous transport with log-correction in $d = 2$

no numerics yet

- ☑ we discovered the "$2/3$ law" for the variance-to-mean ratio of conductance, and propose that it is a signature of a universality class of normal transport induced by random current matching

illustration by John Tenniel

Lorentz mirror model

Hierarchical model

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Lorentz mirror model in $d = 3$: numerics

summary

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main theorem: normal transport

theorem: let $d \geq 3$. for sufficiently large A_0 there exist positive constants C_d, C'_d such that

$$C_d \frac{A_n}{L_n} - 1 \leq \mu_n \leq C'_d \frac{A_n}{L_n}$$

for all n. if $A_0 \gg 1$, one has $C_d \simeq C'_d \simeq 1$ and hence $\mu_n \simeq A_n/L_n$ for all n.

for $d = 3$, $A_0 \geq 2$ suffices
$C_3 = 1 - 5/(4A_0)$, $C'_3 = 1 + 1/(3A_0)$

normal transport from non-chaotic deterministic motion in a quenched random environment

proposition: let $d = 1$ and assume $A_0 \gg 1$. then $\mu_n \simeq A_0/L_n$ as long as μ_n is larger than $O(1)$

upper bound

mean conductance $\mu_n = \sum_\ell \ell \, P_n(\ell)$

recursion $P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2d}} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$

$$\mu_n = \sum_{\ell_1,\ldots,\ell_{2d}} \boxed{\sum_\ell \ell \, K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R})} \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$\tilde{\mu}(\ell_\mathrm{L}, \ell_\mathrm{R}) = \frac{\ell_\mathrm{L}\,\ell_\mathrm{R}}{\ell_\mathrm{L} + \ell_\mathrm{R} + 1} \leq \frac{\ell_\mathrm{L} + \ell_\mathrm{R}}{4} + \frac{1}{3}$$

$$\mu_n \leq \sum_{\ell_1,\ldots,\ell_{2d}} \Big(\frac{1}{4} \sum_{j=1}^{2^d} \ell_j + \frac{1}{3} \Big) \prod_{j=1}^{2^d} P_{n-1}(\ell_j) = 2^{d-2} \mu_{n-1} + \frac{1}{3}$$

for $d \neq 2$ $\mu_n + C''_d \leq 2^{d-2}(\mu_{n-1} + C''_d)$

$\leq 2^{(d-2)n}(\mu_0 + C''_d)$ **with** $C''_d = \{3(2^{d-2} - 1)\}^{-1}$

for $d = 2$ $\mu_n \leq \dfrac{n}{3} + \mu_0$

consistent with the log-correction $\mu_n \simeq n/12$

lower bound

mean of the inverse $\eta_n = \sum_\ell (\ell+1)^{-1} P_n(\ell)$

$$u_n = \sum_{\ell_1,\ldots,\ell_{2^{d-1}}} \Big(\sum_{j=1}^{2^{d-1}} \ell_j + 1\Big)^{-1} \prod_{j=1}^{2^{d-1}} P_n(\ell_j)$$

Jensen's inequality implies $\mu_n \geq \eta_n^{-1} - 1$

recursion $P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$

$$\eta_n = \sum_{\ell_1,\ldots,\ell_{2^d}} \boxed{\sum_\ell (\ell+1)^{-1} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R})} \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$= \tilde{\eta}(\ell_\mathrm{L}, \ell_\mathrm{R})$$

$$\tilde{\eta}(\ell_\mathrm{L}, \ell_\mathrm{R}) = \frac{\ell_\mathrm{L} + \ell_\mathrm{R} - 1}{(\ell_\mathrm{L}+1)(\ell_\mathrm{R}+1)} = \frac{1}{\ell_\mathrm{L}+1} + \frac{1}{\ell_\mathrm{R}+1} - \frac{1}{(\ell_\mathrm{L}+1)(\ell_\mathrm{R}+1)}$$

recursive identity $\eta_n = 2u_{n-1} - (u_{n-1})^2$

for $d = 1$, where $u_n = \eta_n$, $\eta_n \leq 2\eta_{n-1} \leq 2^n \eta_0 = 2^n/(A_0+1)$

$$\mu_n \geq (A_0+1)/L_n - 1$$

suggested by ChatGPT 5.2 Pro!!

for $d \geq 3$, we use further clever concavity argument

summary

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no numerics yet

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illustration by John Tenniel