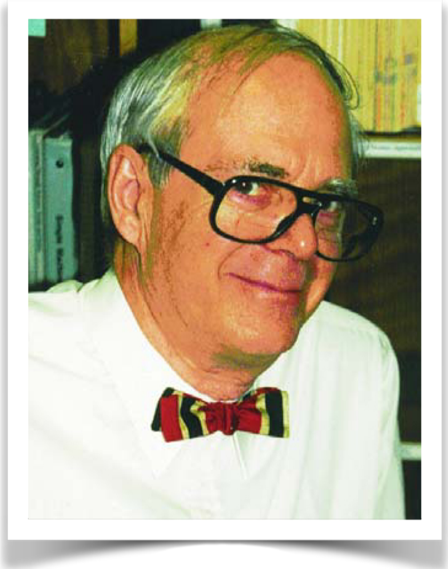
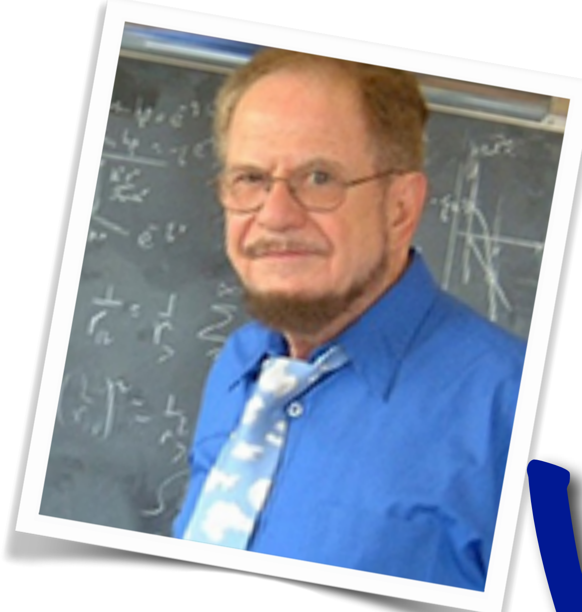




Variations on a Theme by Lieb, Schultz, and Mattis

Unique Gapped Ground States, Symmetry-Protected Topological Phases, Edge States, Spin Pumping, and all that in Quantum Spin Chains

Hal Tasaki

webinar @ YouTube / August 2022

ANNALS OF PHYSICS: 16, 407–466 (1961)

Two Soluble Models of an Antiferromagnetic Chain

ELLIOTT L…

APPENDIX B. NON…
ABSENCE OF AN E…

We prove two exact theor…
for a Heisenberg model wi…
The generalization to longer…
is indicated. A further gener…
of the ordering of excited sta…
in the *Journal of Mathematica*…

THEOREM 1. For a linear c…
magnetic Heisenberg interact…
$S = 0$).

Proof. We first remark that …
shall (7), who proved that th…
exclude the possibility of there…
of which may not be singlets. …

$$H = \sum S^z_i S^z \ldots$$

Next we investigate the nature of the excitation spectrum and prove

THEOREM 2. There is an excited state for the cyclic linear chain with nearest neighbor Heisenberg interactions having vanishingly small excitation energy in the limit that the length of the chain becomes infinite.

Proof. Consider the state

$$\Psi_k = \exp\left(ik \sum nS^z_n\right)\Psi_0 \equiv \mathcal{O}^k \Psi_0 . \qquad \text{(B-11)}$$

We first show that if $k = (2\pi/N) \times$ odd integer, Ψ_k is orthogonal to the ground state. Consider the unitary operator U_x that displaces all the spins by one site cyclically:

$$U_x \mathbf{S}_i U_x^{-1} = \mathbf{S}_{i+1} , \qquad \mathbf{S}_{N+1} = \mathbf{S}_1 . \qquad \text{(B-12)}$$

Because

$$[H, U_x] = 0,$$

if Ψ_0 is an eigenstate of H, so is $U_x\Psi_0$. By the nondegeneracy of Ψ_0

$$U_x\Psi_0 = e^{i\alpha}\Psi_0 . \qquad \text{(B-13)}$$

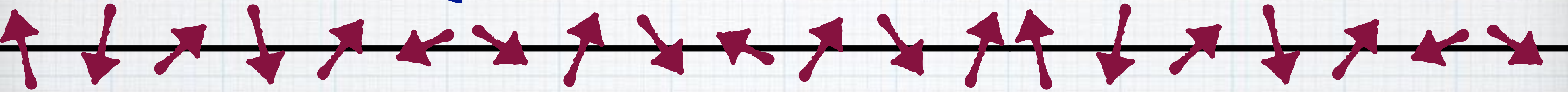
Thus

$$\langle \Psi_0 \mid \Psi_k \rangle = \langle \Psi_0 \mid \mathcal{O}^k \mid \Psi_0 \rangle \qquad \text{(B-14)}$$

- basic setting
- twist operator
  - generalized Lieb-Shultz-Mattis (LSM) theorem
  - topological phase transition in $S=1$ chains
  - edge state in the Haldane phase of $S=1$ chains
  - connecting H_{AKLT} and H_{trivial} without phase transitions
  - Thouless spin pumping in $S=1$ chains
  - classifying loops of Hamiltonians

basic setting

quantum spin chain on $\mathbb{Z}$



spin with $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ on each site $j \in \mathbb{Z}$

$S = 1$ in most part of the talk

spin operators $\boldsymbol{S}_j = (S_j^{\mathrm{x}}, S_j^{\mathrm{y}}, S_j^{\mathrm{z}})$

with $[S_j^{\mathrm{x}}, S_k^{\mathrm{y}}] = i\,\delta_{j,k}\,S_j^{\mathrm{z}}, \ldots$ and $(\boldsymbol{S}_j)^2 = S(S+1)$

act on the local Hilbert space $\mathbb{C}^{2S+1}$

$\mathfrak{A}_{\mathrm{loc}}$ set of all polynomials of S_j^{α}

C^*-algebra $\mathfrak{A} = \overline{\mathfrak{A}_{\mathrm{loc}}}$

Def: A state is a linear function $\rho : \mathfrak{A} \to \mathbb{C}$ such that $\rho(I) = 1$ and $\rho(A^{\dagger}A) \geq 0$ for any $A \in \mathfrak{A}$

$\rho(A)$ is the expectation value of A in the sate ρ

Hamiltonian

formal expression

$$H = \sum_{j\in\mathbb{Z}} h_j$$

local Hamiltonian $h_j = h_j^\dagger \in \mathfrak{A}_{\mathrm{loc}}$

- ☑ **short-ranged**

  h_j acts only on spins at $k \in \{j, \ldots, j+r_0\}$

- ☑ **translation invariant**

  h_j is the translation of h_0

- ☑ U(1) **invariant** — essential assumption

  h_j is invariant under an arbitrary uniform rotation about the z-axis

$$e^{i\theta\sum_{k\in\{j,\ldots,j+r_0\}} S_k^{\mathrm{z}}}\, h_j\, e^{-i\theta\sum_{k\in\{j,\ldots,j+r_0\}} S_k^{\mathrm{z}}} = h_j$$

example: Heisenberg antiferromagnetic chain

$$h_j = \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

unique gapped ground state (g.s.)

Def: A state ω is a g.s. iff $\omega(V^\dagger[H,V]) \geq 0$ for any $V \in \mathfrak{A}_{\text{loc}}$

one cannot lower the energy by a local modification with V

when $\omega(A) = \langle \Phi_{\text{GS}} | A | \Phi_{\text{GS}} \rangle$

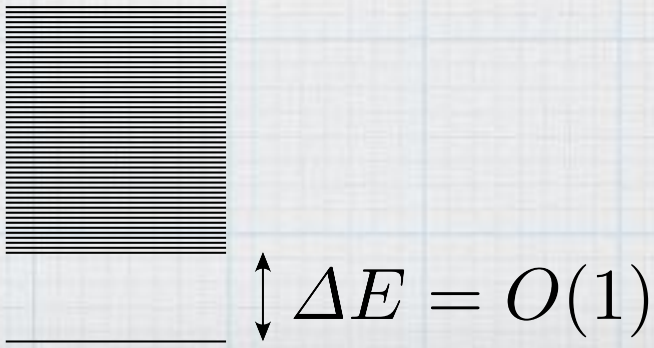
$$\frac{\langle \Phi_{\text{GS}} | V^\dagger H V | \Phi_{\text{GS}} \rangle}{\langle \Phi_{\text{GS}} | V^\dagger V | \Phi_{\text{GS}} \rangle} \geq E_{\text{GS}}$$

a unique g.s. accompanied by a nonzero energy gap

Def: A g.s. ω is a unique gapped g.s. iff it is the only g.s. and $\exists \gamma > 0$ s.t. $\omega(V^\dagger[H,V]) \geq \gamma\, \omega(V^\dagger V)$ for $\forall V \in \mathfrak{A}_{\text{loc}}$ with $\omega(V) = 0$. The energy gap ΔE of ω is the largest γ

$$\frac{\langle \Phi_{\text{GS}} | V^\dagger H V | \Phi_{\text{GS}} \rangle}{\langle \Phi_{\text{GS}} | V^\dagger V | \Phi_{\text{GS}} \rangle} \geq E_{\text{GS}} + \gamma$$

whenever $\langle \Phi_{\text{GS}} | V | \Phi_{\text{GS}} \rangle = 0$



main theme

twist operator

local twist operator

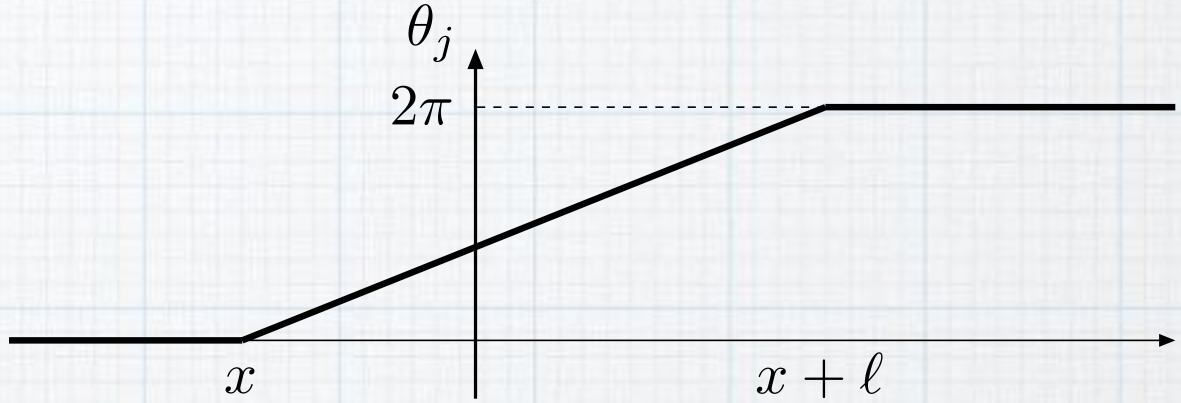
Lieb, Schultz, Mattis 1961, Affleck, Lieb 1987

Bloch 1949

$x \in \mathbb{R},\ \ell > 0$

$$U_{x,\ell} = \exp[-i \textstyle\sum_{j\in[x,x+\ell]\cap\mathbb{Z}} \theta_j\, S_j^{\mathrm{z}}]$$

$$\theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \le j \le x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$

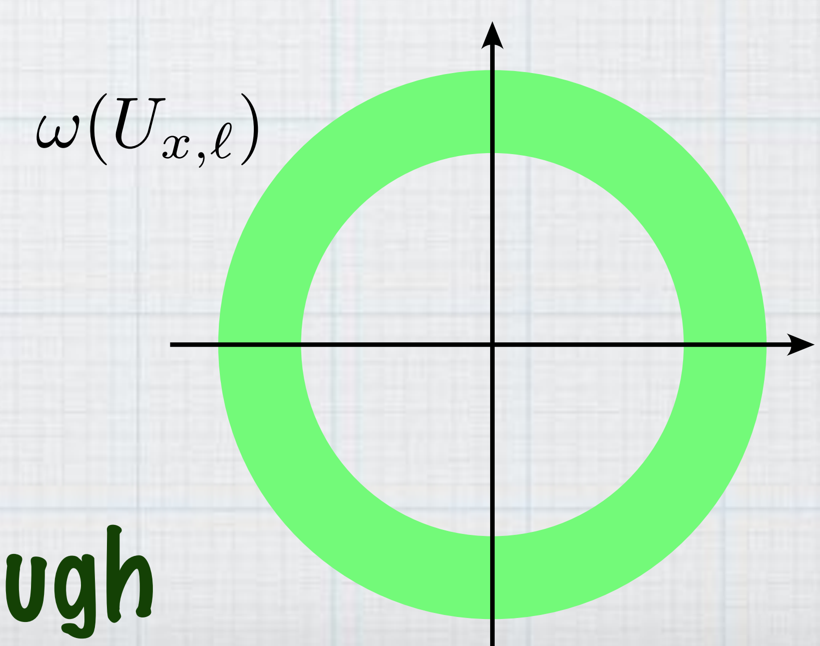


Lemma: Let ω be a g.s. Then for any x and $\ell \ge \ell_0$ one has $0 \le \omega(U_{x,\ell}^\dagger[H, U_{x,\ell}]) \le \frac{C}{\ell}$ where C and ℓ_0 are constants

$\omega(U_{x,\ell}) = 0$ → (energy gap of ω) $\le \frac{C}{\ell}$ gapless!

the original logic in Lieb, Schultz, Mattis and Affleck, Lieb

Lemma: Let ω be a unique gapped g.s. with energy gap $\Delta E > 0$. Then one has $1 - \frac{C}{\ell\,\Delta E} \le |\omega(U_{x,\ell})|^2 \le 1$

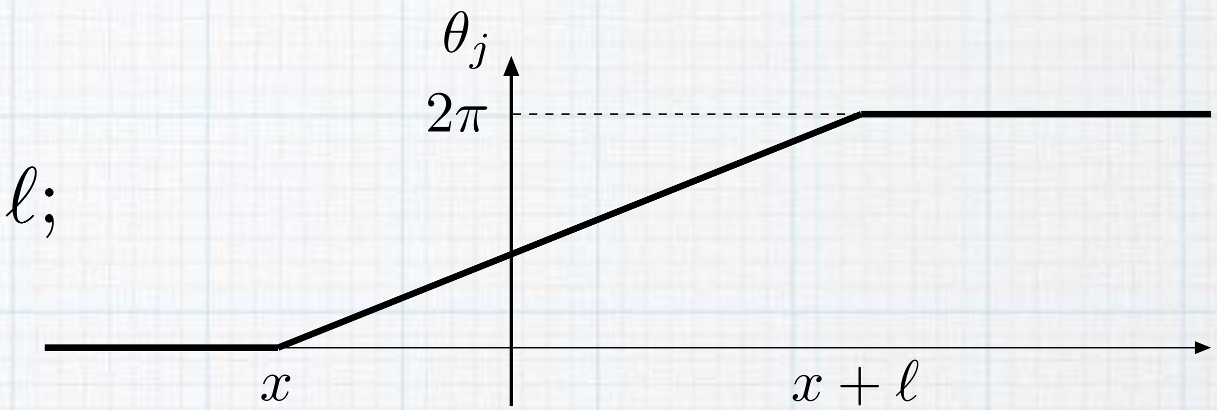


$|\omega(U_{x,\ell})| \simeq 1$ if ℓ is large enough

generalized Lieb-Shultz-Mattis (LSM) theorem

a modification to the twist operator

$$\theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \le j \le x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$



$$U_{x,\ell} = \exp\Big[-i \sum_{j\in[x,x+\ell]\cap\mathbb{Z}} \theta_j\, S_j^{\mathrm{z}}\Big]$$

$$\tilde{U}_{x,\ell} = \exp\Big[-i \sum_{j\in[x,x+\ell]\cap\mathbb{Z}} \theta_j\, S\Big]\, U_{x,\ell} = \exp\Big[-i \sum_{j\in\mathbb{Z}} \theta_j (S_j^{\mathrm{z}} + S)\Big]$$

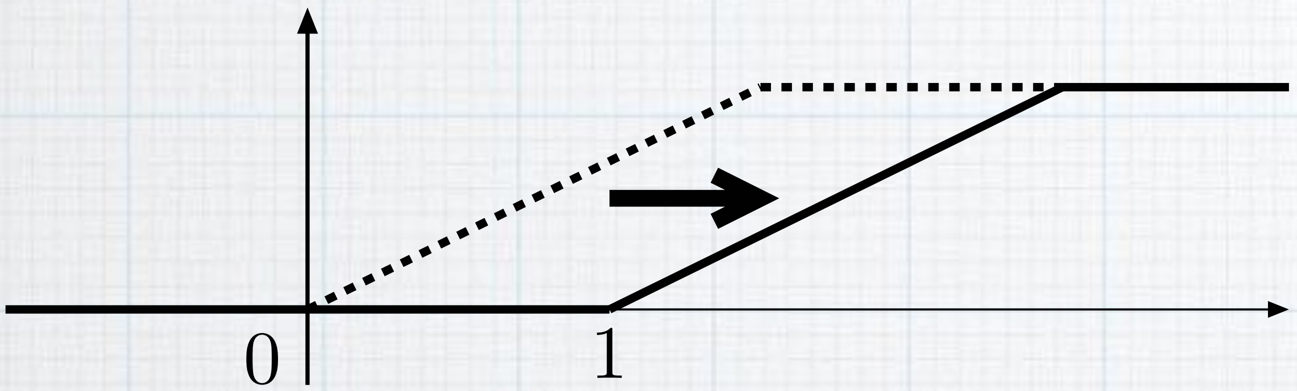
$$\exp[-i2\pi(S_j^{\mathrm{z}} + S)] = 1$$

$\tilde{U}_{x,\ell}$ **is continuous in x and ℓ**

the modification is necessary only for $S = \frac{1}{2}, \frac{3}{2}, \ldots$

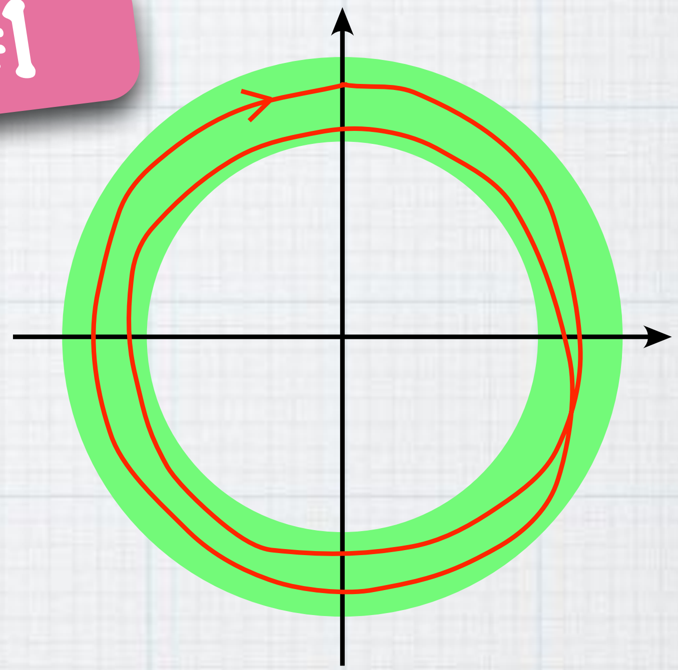
winding number

suppose that ω is a unique gapped g.s. → translation invariant



$$\omega(\tilde{U}_{0,\ell}) = \omega(\tilde{U}_{1,\ell})$$

variation #1



$\omega(\tilde{U}_{x,\ell})$ is continuous in $x \in [0,1]$

$|\omega(\tilde{U}_{x,\ell})| \simeq 1$ for sufficiently large ℓ

well-defined winding number $\nu_\omega \in \mathbb{Z}$

topological index

invariant under continuous modification of unique gapped g.s.

Lemma: $\nu_\omega = \omega(S_j^z + S)$

filling factor

$$\tilde{U}_{x,\ell} = \exp[i2\pi x \tfrac{1}{\ell} \textstyle\sum_{j=1}^{\ell}(S_j^z + S)]\,\tilde{U}_{0,\ell}$$

$$\omega(\tilde{U}_{x,\ell}) \simeq e^{i2\pi x\,\omega(S_j^z+S)}\omega(\tilde{U}_{0,\ell})$$

generalized LSM theorem

quantum spin chain with a short-ranged $\mathrm{U}(1)$ invariant translation-invariant Hamiltonian

ω is a unique gapped g.s. → $\omega(S_j^{\mathrm{z}} + S) \in \mathbb{Z}$

Theorem Oshikawa, Yamanaka, Affleck 1997, Tasaki 2018

$\omega(S_j^{\mathrm{z}} + S) \notin \mathbb{Z}$ → ω cannot be a unique gapped g.s.

for models where a unique g.s. satisfies $\omega(S^{\mathrm{z}}) = 0$

Theorem Affleck, Lieb 1986

$S = \frac{1}{2}, \frac{3}{2}, \ldots$ → there can be no unique gapped g.s.

"half" of the Haldane conjecture

Haldane 1981, 1983, 1983

the Heisenberg AF chain has a unique gappless g.s if $S = \frac{1}{2}, \frac{3}{2}, \ldots$, and a unique gapped g.s. if $S = 1, 2, \ldots$

generalized LSM theorem

quantum spin chain with a short-ranged $\mathrm{U}(1)$ invariant translation-invariant Hamiltonian

ω is a unique gapped g.s. → $\omega(S_j^{\mathrm{z}} + S) \in \mathbb{Z}$

Theorem Oshikawa, Yamanaka, Affleck 1997, Tasaki 2018

$\omega(S_j^{\mathrm{z}} + S) \notin \mathbb{Z}$ → ω cannot be a unique gapped g.s.

for models where a unique g.s. satisf

Theorem

$S = \frac{1}{2}$ … pped g.s.

Lieb-Schultz-Mattis theorem:
a no-go theorem that states that a certain quantum many-body system cannot have a unique gapped g.s.

… of the Haldane conjecture

Haldane 1981, 1983, 1983

the Heisenberg AF chain has a unique gapless g.s if $S = \frac{1}{2}, \frac{3}{2}, \ldots$, and a unique gapped g.s. if $S = 1, 2, \ldots$

$S = 1$ in the rest of the talk

topological phase transition in $S = 1$ chains

models with unique gapped g.s.

Theorem Affleck, Lieb 1986

$S = \frac{1}{2}, \frac{3}{2}, \ldots$ → **there can be no unique gapped g.s.**

for $S = 1$, $\exists$quantum spin chains with $\mathrm{U}(1)$ and translation invariant Hamiltonians which have a unique gapped g.s.

☑ **trivial model** $H_{\text{trivial}} = \sum_{j\in\mathbb{Z}} (S_j^{\mathrm{z}})^2$

g.s. $= \bigotimes_{j\in\mathbb{Z}} |0\rangle_j \qquad S_j^{\mathrm{z}}|0\rangle_j = 0$

invariant under any rotation about the z-axis and $S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$

non-interacting model with $\mathrm{U}(1) \rtimes \mathbb{Z}_2$ symmetry

☑ **AKLT model** $H_{\text{AKLT}} = \sum_{j\in\mathbb{Z}} \{\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\}$

g.s. = 

Affleck, Kennedy, Lieb, Tasaki 1987, 1988, Matsui 1997

antiferromagnetic model with $\mathrm{SU}(2)$ symmetry

models with unique gapped g.s.

Theorem Affleck, Lieb 1986

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invariant under any rotation about the z-axis and $S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$

g.s. $= \bigotimes_{j\in\mathbb{Z}} |0\rangle_j \qquad S_j^{\mathrm{z}}|0\rangle_j = 0$

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g.s. = 

closely reltaed to the Haldane conjecture

Affleck, Kennedy, Lieb, Tasaki 1987, 1988, Matsui 1997

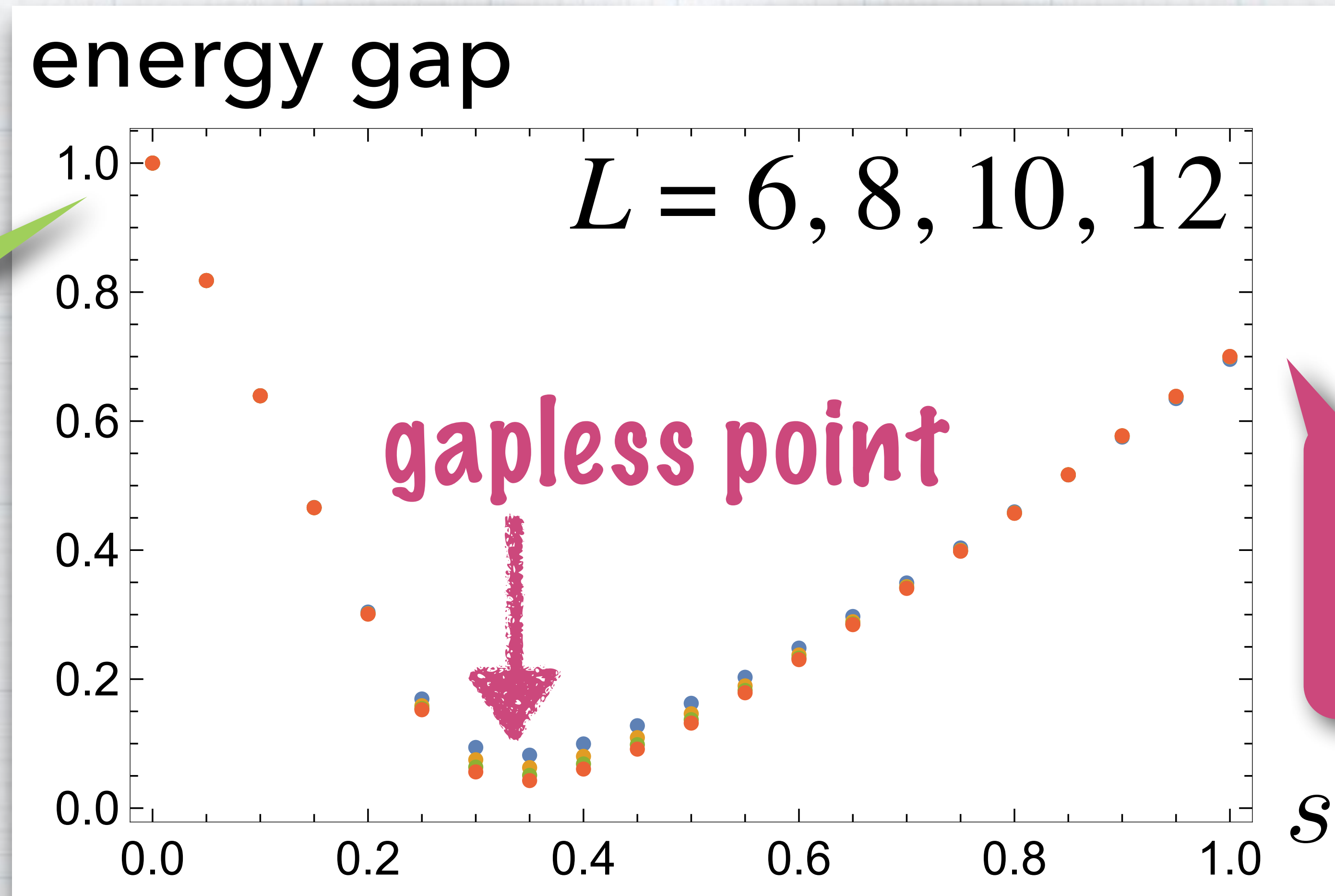
antiferromagnetic model with $\mathrm{SU}(2)$ symmetry

Interpolating model

$$H_s = s\,H_{\mathrm{AKLT}} + (1-s)\,H_{\mathrm{trivial}} \qquad s \in [0,1] \qquad S=1$$

a unique gapped g.s. at $s = 0{,}1$

trivial gap



Haldane gap

numerical results by Hosho Katsura

there is a phase transition at intermediate s !!

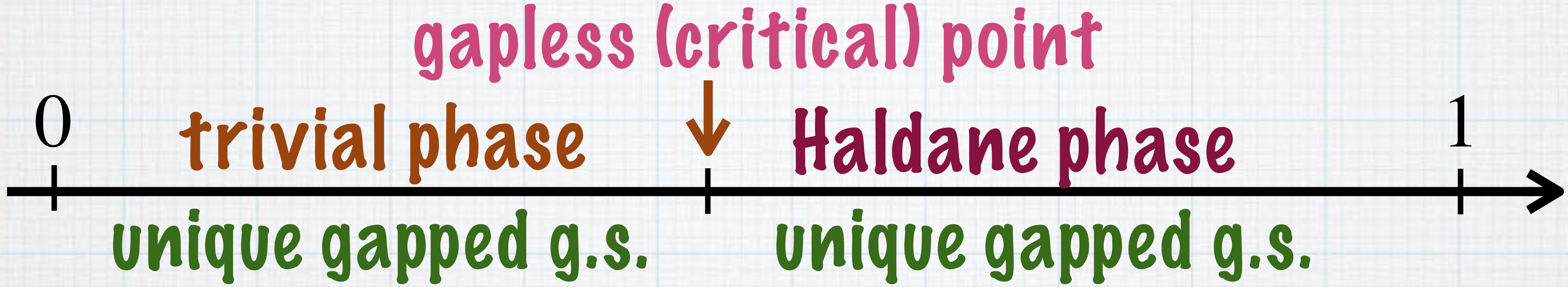
gapless (critical) point

0 trivial phase ↓ Haldane phase 1 s

unique gapped g.s. unique gapped g.s.

topological phase transition

$H_s = s\,H_{\text{AKLT}} + (1-s)\,H_{\text{trivial}}$ $s \in [0,1]$ $S = 1$



how can one distinguish between the two "phases"?

is there really a phase transition?

hidden antiferromagnetic order den Nijs, Rommelse 1989

the emergence of edge states

Kennedy 1990, Hagiwara, Katsumata, Affleck, Halperin, Renard 1990

hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking Kennedy, Tasaki 1992

theory of symmetry protected topological (SPT) phases

Gu, Wen 2009, Pollmann, Turner, Berg, Oshikawa 2010, 2012

proof of the existence of a phase transition

Tasaki 2018, Ogata 2018

twist operator as an "order parameter"

$H_s = s\,H_{\rm AKLT} + (1-s)\,H_{\rm trivial}$ $\quad s \in [0,1] \quad S=1$

gapless (critical) point

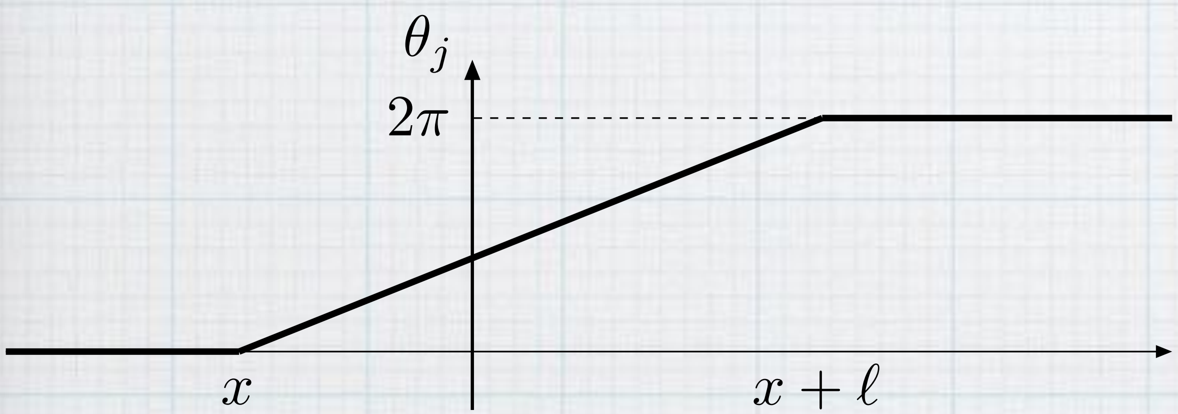
0 trivial phase ↓ Haldane phase 1 s

unique gapped g.s. unique gapped g.s.

how can one distinguish between the two "phases"?

$U_{x,\ell} = \exp[-i \sum_{j\in\mathbb{Z}} \theta_j S^{\rm z}_j]$

$$\theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \le j \le x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$



$U_{x,\ell}$ continuous in x and ℓ

variation #2

$\omega_{\rm trivial}(U_{x,\ell}) = 1$

$\omega_{\rm AKLT}(U_{x,\ell}) \simeq -1$ for sufficiently large ℓ

$\omega_s(U_{x,\ell})$ can be used as an "order parameter"

Nakamura, Todo 2002

twist operator and $\mathbb{Z}_2$ topological index

$$H_s = s\,H_{\mathrm{AKLT}} + (1-s)\,H_{\mathrm{trivial}} \qquad s\in[0,1] \qquad S=1$$

$\mathrm{U}(1)\rtimes\mathbb{Z}_2$ symmetry

H_s is invariant under $S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$

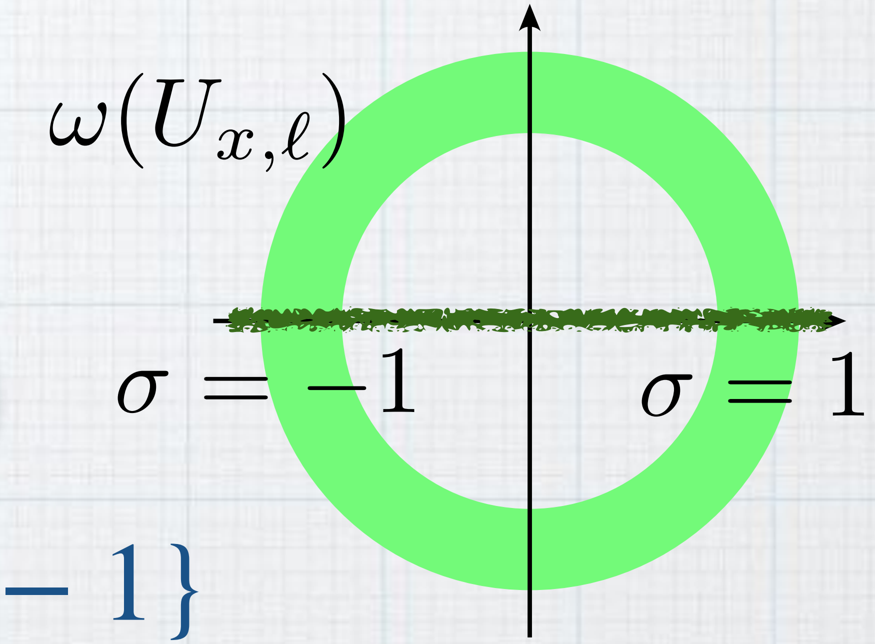
$$U_{x,\ell} = \exp[-i\sum_{j\in\mathbb{Z}}\theta_j S_j^{\mathrm{z}}] \to \exp[-i\sum_{j\in\mathbb{Z}}\theta_j(-S_j^{\mathrm{z}})] = U_{x,\ell}^\dagger$$

if ω_s is a unique gapped g.s. of H_s

$$\omega_s(U_{x,\ell}) = \omega_s(U_{x,\ell}^\dagger)\in\mathbb{R}$$

$$\omega_s(U_{x,\ell}) \simeq \pm 1$$

variation #3



well-defined topological index $\sigma_s\in\{1,-1\}$

such that $\sigma_0=1$ and $\sigma_1=-1$

Theorem Tasaki 2018

there is $s\in(0,1)$ at which ω_s is either not a unique gapped g.s. or exhibits discontinuity

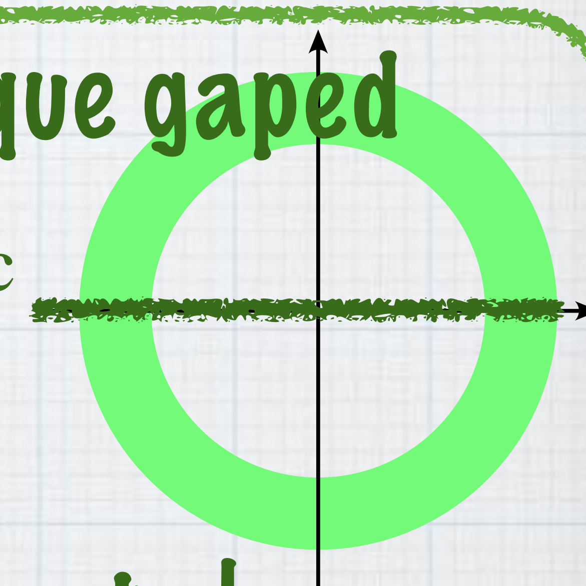
twist operator and $\mathbb{Z}_2$ topological index

$H_s = s\,H_{\mathrm{AKLT}} + (1-s)\,H_{\mathrm{trivial}}$ $s \in [0,1]$ $S=1$

Theorem Tasaki 2018

There is $s \in (0,1)$ at which ω_s is either not a unique gapped g.s. or exhibits discontinuity

proof: suppose that for $\forall s \in [0,1]$, ω_s is a unique gaped g.s. and $\omega_s(A)$ is continuous in s for $\forall A \in \mathfrak{A}_{\mathrm{loc}}$



then $\omega_s(U_{x,\ell}) \simeq \pm 1$ for $\forall s \in [0,1]$ if ℓ is sufficiently large

since $\omega_s(U_{x,\ell})$ is continuous in s, its sign σ_s cannot change

Ogata 2018 a complete theory of SPT phases that only requires the minimum symmetry, e.g., $\mathbb{Z}_2 \times \mathbb{Z}_2$

Tasaki's theorem is elementary, but requires larger symmetry, e.g., $\mathrm{U}(1) \rtimes \mathbb{Z}_2$

edge state in the Haldane phase of $S = 1$ chains

edge state in the Haldane phase

effective $S = \frac{1}{2}$

spin chain of $S = 1$

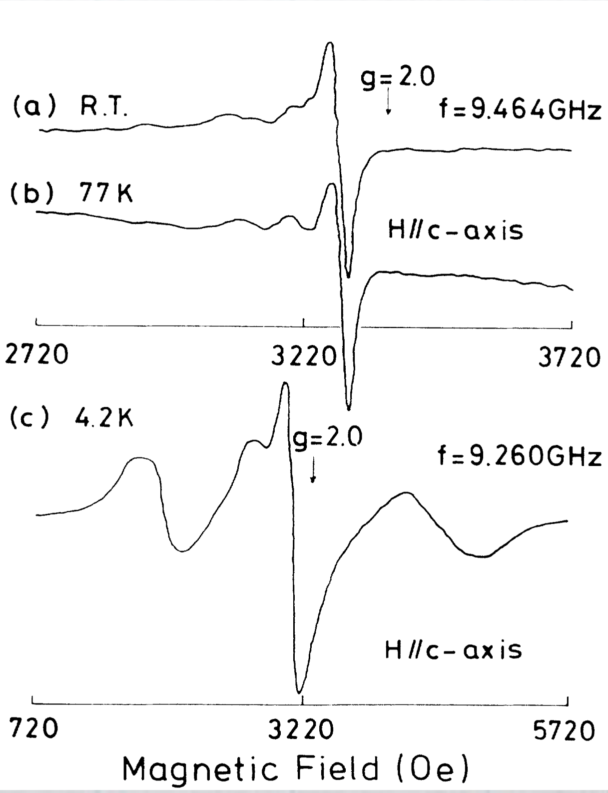
H_{AKLT} on the infinite chain has a unique gapped g.s.

H_{AKLT} on the half-infinite chain has doubly degenerate g.s.

Affleck, Kennedy, Lieb, Tasaki 1988

the emergence of the effective $S = \frac{1}{2}$ degree of freedom at the edge is a universal property of the Haldane phase

Kennedy 1990
Hagiwara, Katsumata, Affleck, Halperin, Renard 1990



the existence of an edge state

$H_s = s\, H_{\mathrm{AKLT}} + (1-s)\, H_{\mathrm{trivial}}$ $\quad s \in [0,1] \qquad S=1$

$$H_s = \sum_{j\in\mathbb{Z}} h_j^{(s)}$$

$$h_j^{(s)} = s\,\{\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1} + \tfrac{1}{3}(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})^2\} + (1-s)\,(S_j^{\mathrm{z}})^2$$

$H'_s = \sum_{j\in\mathbb{Z}_+} h_j^{(s)}$ Hamiltonian for spin system on $\mathbb{Z}_+$

Theorem Tasaki 2021

Suppose that H_s has a unique gapped g.s. with $\sigma_s = -1$, and let ω'_s be an arbitrary g.s. of H'_s

For arbitrary $\varepsilon > 0$, there exists a unitary U_ε such that $\omega'_s(U_\varepsilon) = 0$ and $\omega'_s(U_\varepsilon^\dagger[H'_s, U_\varepsilon]) \le \varepsilon$

Moreover, U_ε is local and acts near the edge of $\mathbb{Z}_+$

$U_\varepsilon|\Phi'_{\mathrm{GS}}\rangle$ is orthogonal to $|\Phi'_{\mathrm{GS}}\rangle$, and has excitation energy $\le \varepsilon$

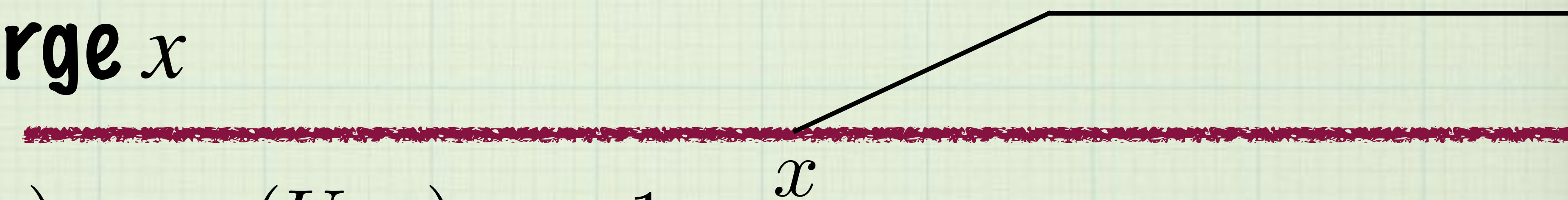
proof

variation #4

$$U_{x,\ell} = \exp[-i \sum_{j\in\mathbb{Z}} \theta_j S_j^{\mathrm{z}}]$$

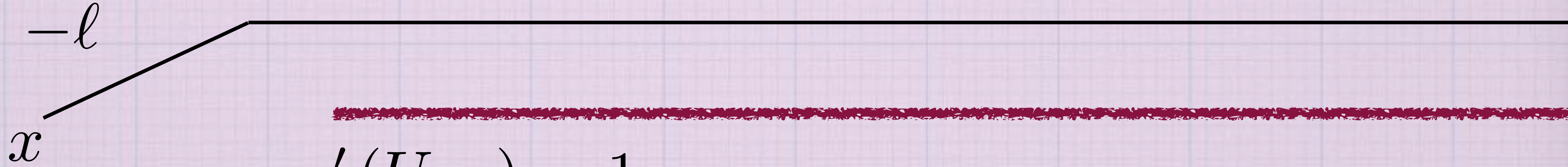
$$\theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \le j \le x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$

for sufficiently large x



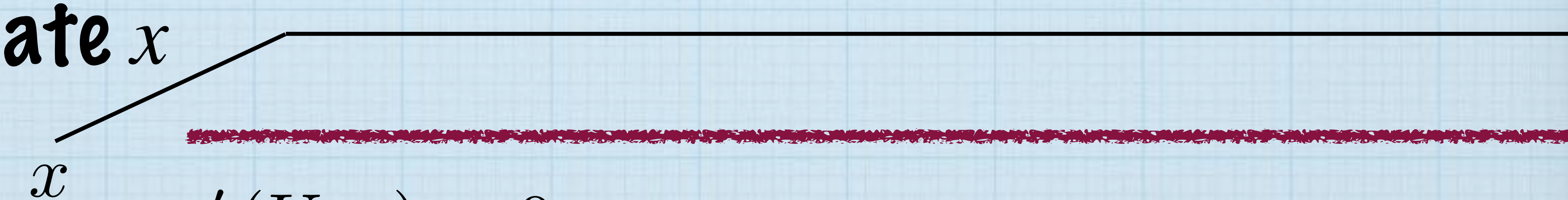
$$\omega_s'(U_{x,\ell}) \simeq \omega_s(U_{x,\ell}) \simeq -1$$

for $x \le -\ell$



$$\omega_s'(U_{x,\ell}) = 1$$

at intermediate x



$$\omega_s'(U_{x,\ell}) = 0$$

$$\omega_s'(U_{x,\ell}) = 0$$

$$\omega_s'(U_{x,\ell}^\dagger [H_s', U_{x,\ell}]) \le \frac{C}{\ell}$$

→ there is an excited state with the excitation energy $\le \frac{C}{\ell}$

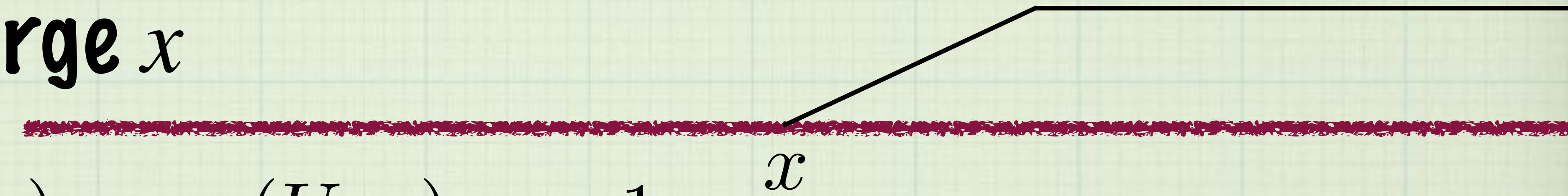
proof

variation #4

$$U_{x,\ell} = \exp[-i \textstyle\sum_{j\in\mathbb{Z}} \theta_j S_j^{\mathrm{z}}]$$

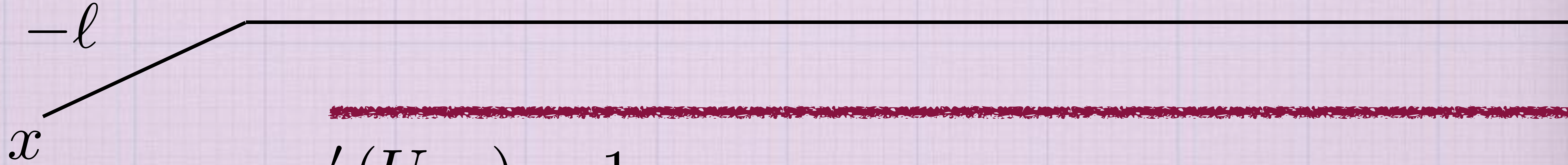
$$\theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \le j \le x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$

for sufficiently large x



$$\omega'_s(U_{x,\ell}) \simeq \omega_s(U_{x,\ell}) \simeq -1$$

for $x \le -\ell$



$$\omega'_s(U_{x,\ell}) = 1$$

at intermediate x

the original idea of Lieb, Schultz, and Mattis

$$\omega'_s(U_{x,\ell}) = 0$$

$$\omega'_s(U^\dagger_{x,\ell}[H'_s, U_{x,\ell}]) \le \frac{C}{\ell}$$

→ **there is an excited state with the excitation energy $\le \frac{C}{\ell}$**

connecting H_{AKLT} and H_{trivial} without phase transitions

SPT phases in $S = 1$ chains

$H_s = s\, H_{\mathrm{AKLT}} + (1 - s)\, H_{\mathrm{trivial}}$ $s \in [0, 1]$ $S = 1$

0 trivial phase Haldane phase 1

the picture of symmetry protected topological (SPT) phases

Pollmann, Turner, Berg, Oshikawa 2010, 2012, Ogata 2020, 2022

more generally, there always is a phase transition if

$H_0 = H_{\mathrm{trivial}}$, $H_1 = H_{\mathrm{AKLT}}$, and H_s has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

π rotations about the x and z axes

one may connect the two models without phase transition if the symmetry is not respected

examples Bachmann, Nachtergaele 2012, 2014, Maekawa, Tasaki 2022

general theory within MPS Ogata 2017

interpolating the g.s. of H_{trivial} and H_{AKLT}

Maekawa, Tasaki 2022
Katsura (private comm.)

the g.s. of H_{AKLT} on the open chain $\{1,\ldots,N\}$

$$|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle=\Bigl(\bigotimes_{j=1}^{N}\mathcal{S}_j\Bigr)\Bigl\{|\uparrow\rangle_{1,\mathrm{L}}\otimes\Bigl(\bigotimes_{j=1}^{N-1}\bigl(|\uparrow\rangle_{j,\mathrm{R}}|\downarrow\rangle_{j+1,\mathrm{L}}-|\downarrow\rangle_{j,\mathrm{R}}|\uparrow\rangle_{j+1,\mathrm{L}}\bigr)\Bigr)\otimes|\downarrow\rangle_{N,\mathrm{R}}\Bigr\}$$

LSM-like twist operator with imaginary angles

$$V_s=s^{-\sum_{j=1}^{N}j\,S_j^{\mathrm{z}}}=\exp\bigl[-\textstyle\sum_{j=1}^{N}\theta_j S_j^{\mathrm{z}}\bigr]\qquad\theta_j=j\,\log s$$

twisted state

$s\in(0,1]$

variation #5

$$|\Phi_s\rangle=\frac{V_s|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle}{\|V_s|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle\|}$$

$$=C_s\Bigl(\bigotimes_{j=1}^{N}\mathcal{S}_j\Bigr)\Bigl\{|\uparrow\rangle_{1,\mathrm{L}}\otimes\Bigl(\bigotimes_{j=1}^{N-1}\bigl(s\,|\uparrow\rangle_{j,\mathrm{R}}|\downarrow\rangle_{j+1,\mathrm{L}}-|\downarrow\rangle_{j,\mathrm{R}}|\uparrow\rangle_{j+1,\mathrm{L}}\bigr)\Bigr)\otimes|\downarrow\rangle_{N,\mathrm{R}}\Bigr\}$$

interpolating the g.s. of H_{trivial} and H_{AKLT}

Maekawa, Tasaki 2022

twisted state $s \in (0,1]$

$$|\Phi_s\rangle = \frac{V_s|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle}{\|V_s|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle\|}$$

$$= C_s\Bigl(\bigotimes_{j=1}^{N}\mathcal{S}_j\Bigr)\Bigl\{|{\uparrow}\rangle_{1,\mathrm{L}}\otimes\Bigl(\bigotimes_{j=1}^{N-1}\bigl(s\,|{\uparrow}\rangle_{j,\mathrm{R}}|{\downarrow}\rangle_{j+1,\mathrm{L}}-|{\downarrow}\rangle_{j,\mathrm{R}}|{\uparrow}\rangle_{j+1,\mathrm{L}}\bigr)\Bigr)\otimes|{\downarrow}\rangle_{N,\mathrm{R}}\Bigr\}$$

$|\Phi_1\rangle = (\mathrm{const})\,|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle$ **a g.s. of** H_{AKLT}

$|\Phi_0\rangle = \bigotimes_{j=1}^{N}|0\rangle_j$ **the unique g.s. of** H_{trivial}

$\tilde{\omega}_s$ **the infinite volume limit (on $\mathbb{Z}$) of** $|\Phi_s\rangle$

translation invariant

Theorem: There is a continuous family $\tilde{H}_s$ (with $s \in [0,1]$) of Hamiltonians on $\mathbb{Z}$ with $\tilde{H}_1 = H_{\mathrm{AKLT}}$ and $\tilde{H}_0 = H_{\mathrm{trivial}}$ such that $\tilde{\omega}_s$ is a unique gapped g.s. of $\tilde{H}_s$ for $s \in [0,1]$.

interpolating the g.s. of H_{trivial} and H_{AKLT}

Maekawa, Tasaki 2022

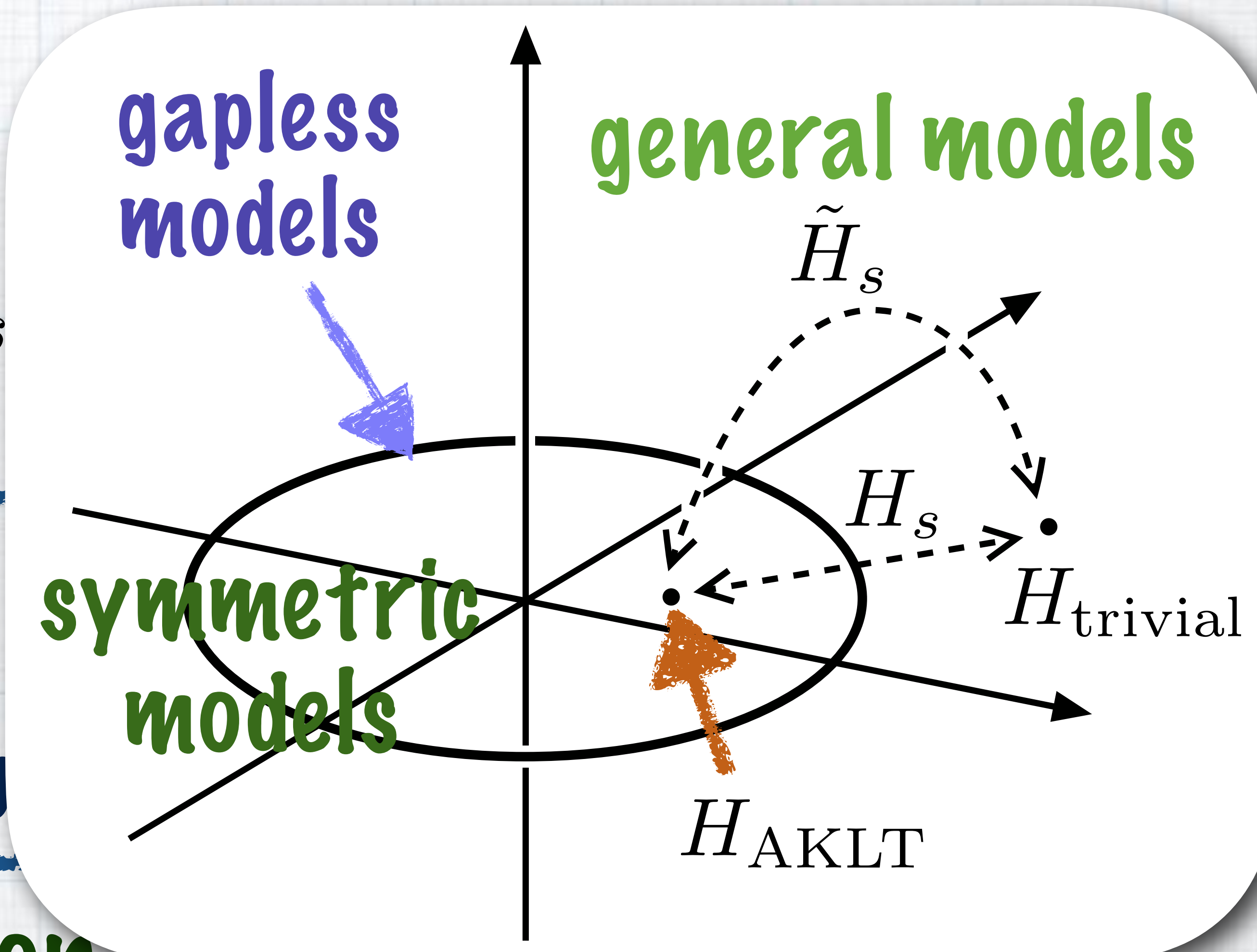
twisted state $s \in (0,1]$

$$|\Phi_s\rangle = \frac{V_s|\Phi^{\uparrow,\downarrow}_{\text{AKLT}}\rangle}{\|V_s|\Phi^{\uparrow,\downarrow}_{\text{AKLT}}\rangle\|}$$

$$= C_s\Bigl(\bigotimes_{j=1}^{N}\mathcal{S}_j\Bigr)\Bigl\{|\uparrow\rangle_{1,\text{L}}\otimes\Bigl(\bigotimes_{j=1}^{N-1}(s \ldots \text{R}\Bigr\}$$

$$|\Phi_1\rangle = (\text{const})\,|\Phi^{\uparrow,\downarrow}_{\text{AKLT}}\rangle$$

$$|\Phi_0\rangle = \bigotimes_{j=1}^{N}|0\rangle_j$$ **the u**



$\tilde{\omega}_s$ **the infinite volume limit (on $\mathbb{Z}$) of $|\Phi_s\rangle$**

translation invariant

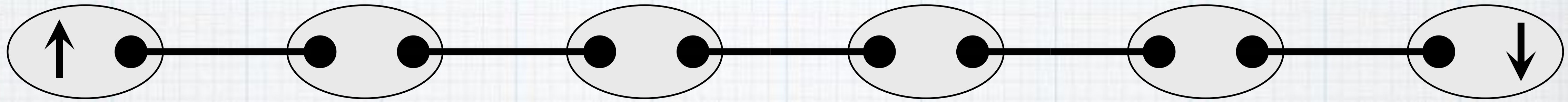
Theorem: There is a continuous family $\tilde{H}_s$ (with $s \in [0,1]$) of Hamiltonians on $\mathbb{Z}$ with $\tilde{H}_1 = H_{\text{AKLT}}$ and $\tilde{H}_0 = H_{\text{trivial}}$ such that $\tilde{\omega}_s$ is a unique gapped g.s. of $\tilde{H}_s$ for $s \in [0,1]$.

Thouless spin pumping in $S = 1$ chains

spin pumping in the previous example

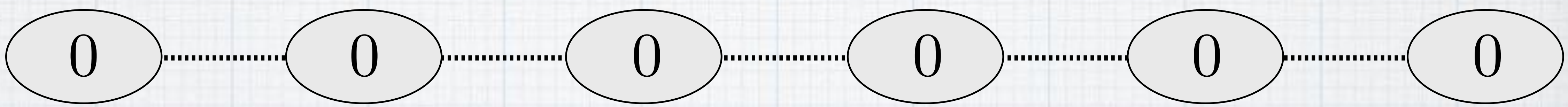
Maekawa, Tasaki 2022

$|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle$



the LSM-like twist with imaginary angles

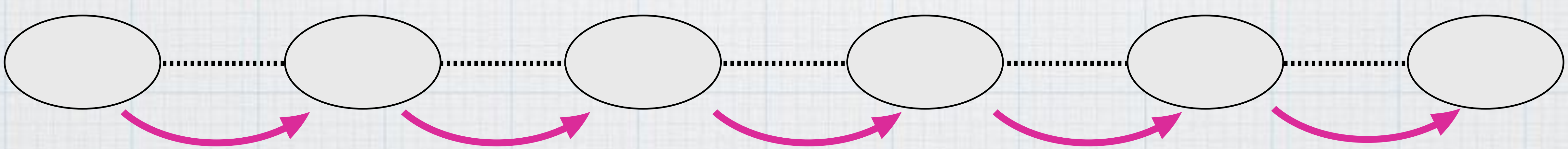
$|\Phi_0\rangle$



the reversed and reflected transformation

$|\Phi^{\downarrow,\uparrow}_{\mathrm{AKLT}}\rangle$

$S^{\mathrm{z}} = +1$ is pumped from left to right!



S^{z}_j may be treated as a local charge since $\sum_j S^{\mathrm{z}}_j$ is conserved

Thouless 1983, Shindou 2005

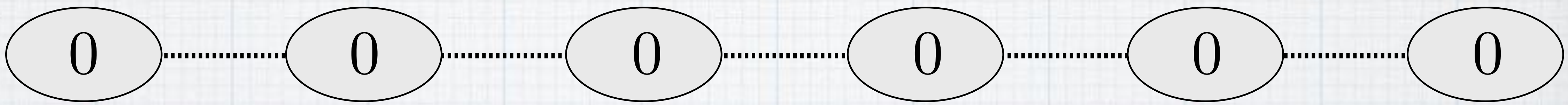
spin pumping in the previous example

Maekawa, Tasaki 2022

$|\Phi^{\uparrow,\downarrow}_{\mathrm{AKLT}}\rangle$

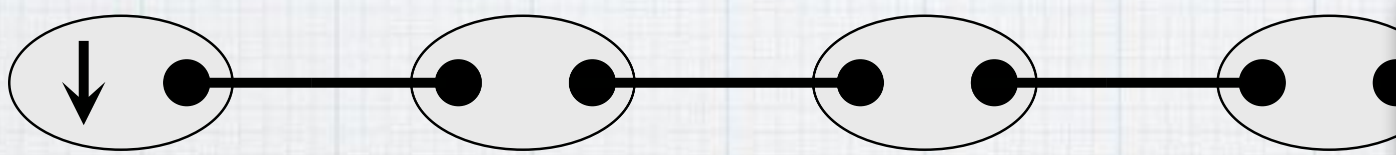
the LSM-like twist with imaginary angles

$|\Phi_0\rangle$



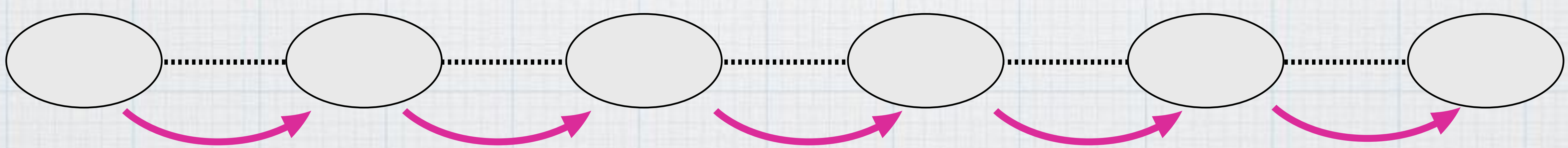
the reversed and reflected transformation

$|\Phi^{\downarrow,\uparrow}_{\mathrm{AKLT}}\rangle$



this is a general phenomenon!

$S^z = +1$ is pumped from left to right!



S^z_j may be treated as a local charge since $\sum_j S^z_j$ is conserved

Thouless 1983, Shindou 2005

pumping in an infinite quantum spin chain

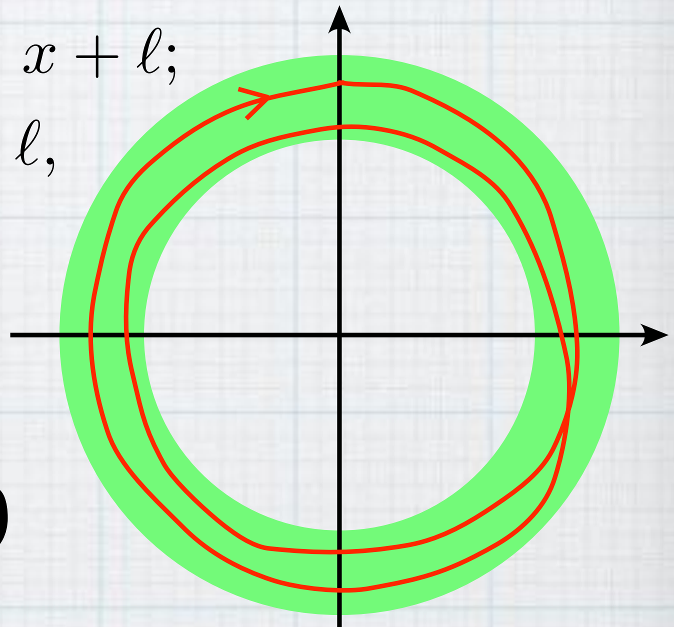
quantum spin chains on $\mathbb{Z}$ with Hamiltonian H_s^{p} ($s \in [0,1]$)

H_s^{p} is $\mathrm{U}(1)$ invariant $\quad H_0^{\mathrm{p}} = H_1^{\mathrm{p}}$

H_s^{p} has a unique gapped g.s. ω_s^{p} with gap $\geq \Delta E_0 > 0$

$\omega_s^{\mathrm{p}}(A)$ is continuous in $s \in [0,1]$ for any $A \in \mathfrak{A}_{\mathrm{loc}}$

$$U_{x,\ell} = \exp[-i \textstyle\sum_{j\in\mathbb{Z}} \theta_j S_j^{\mathrm{z}}] \qquad \theta_j = \begin{cases} 0, & j < x; \\ 2\pi(j-x)/\ell, & x \leq j \leq x+\ell; \\ 2\pi, & j > x+\ell, \end{cases}$$



fix x and sufficiently large ℓ

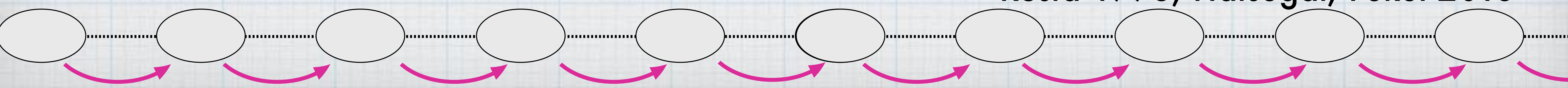
$$|\omega_s^{\mathrm{p}}(U_{x,\ell})| \simeq 1$$

variation #6

$\omega_s^{\mathrm{p}}(U_{x,\ell}) \in \mathbb{C}$ with $s \in [0,1]$ defines a loop

Claim: the corresponding winding number $p \in \mathbb{Z}$ is the amount of S^{z} pumped from left to right in the g.s. when s is slowly changed from 0 to 1

Thouless 1983, Niu, Thouless 1984
Resta 1998, Hatsugai, Fukui 2016



SPT phases and "half-spin pumping"

quantum spin chains on $\mathbb{Z}$ with Hamiltonian H_s^{p} ($s \in [0,1/2]$)

H_s^{p} is $\mathrm{U}(1)$ invariant

H_s^{p} has a unique gapped g.s. ω_s^{p} with gap $\geq \Delta E_0 > 0$

$\omega_s^{\mathrm{p}}(A)$ is continuous in $s \in [0,1/2]$ for any $A \in \mathfrak{A}_{\mathrm{loc}}$

H_0^{p} and $H_{1/2}^{\mathrm{p}}$ are $\mathrm{U}(1) \rtimes \mathbb{Z}_2$ invariant

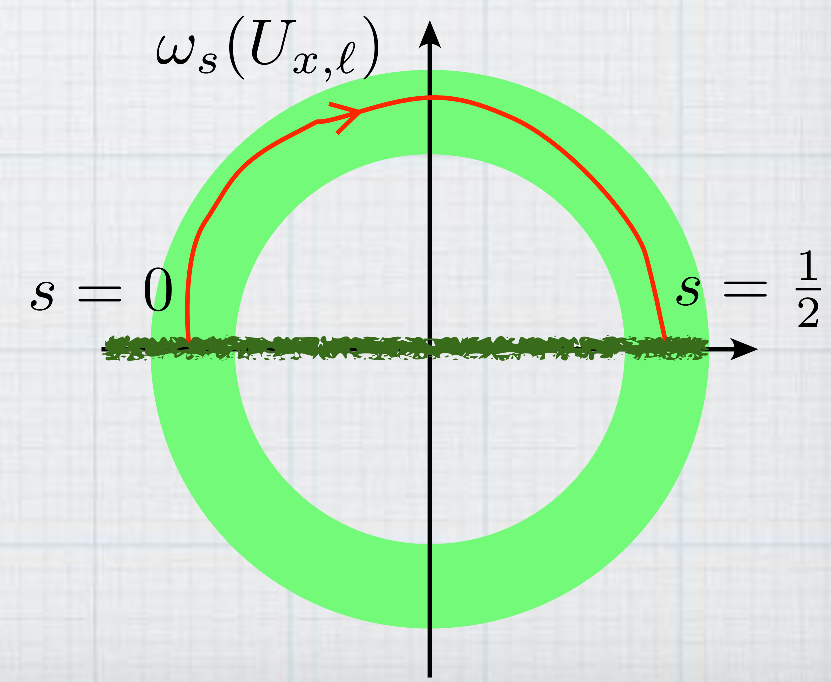
invariant under any rotation about the z-axis and $S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$

$\sigma_0^{\mathrm{p}} = -1$ and $\sigma_{1/2}^{\mathrm{p}} = 1$

Haldane phase

trivial phase

$\omega_0^{\mathrm{p}}(U_{x,\ell}) \simeq -1$ $\qquad$ $\omega_{1/2}^{\mathrm{p}}(U_{x,\ell}) \simeq 1$



the "winding number" is inevitably nonzero! $\quad p_{0\to 1/2} \in \mathbb{Z} + \frac{1}{2}$

SPT phases and spin pumping

quantum spin chains on $\mathbb{Z}$ with Hamiltonian H_s^{p} ($s \in [0,1]$)

H_s^{p} is U(1) invariant $\quad H_0^{\mathrm{p}} = H_1^{\mathrm{p}}$

H_s^{p} has a unique gapped g.s. ω_s^{p} with gap $\geq \Delta E_0 > 0$

$\omega_s^{\mathrm{p}}(A)$ is continuous in $s \in [0,1]$ for any $A \in \mathfrak{A}_{\mathrm{loc}}$

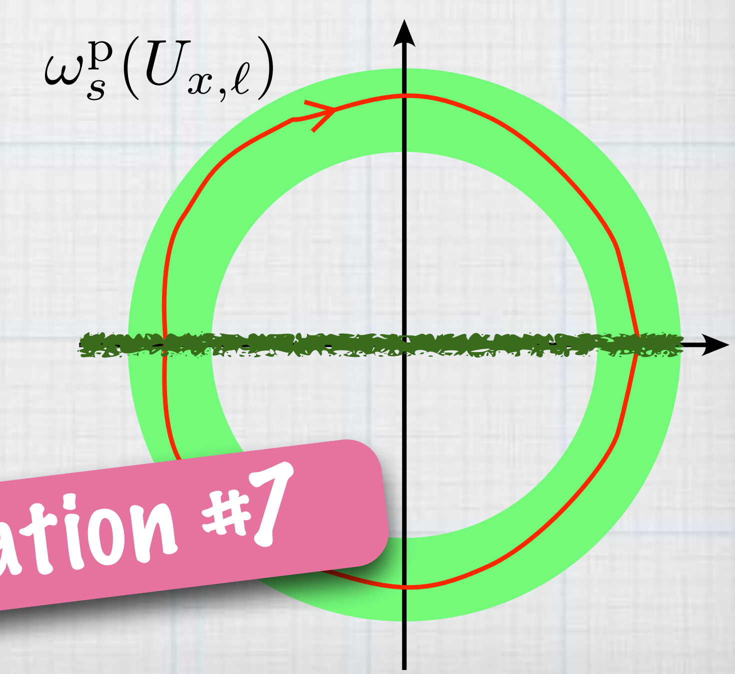
H_0^{p} and $H_{1/2}^{\mathrm{p}}$ are $\mathrm{U}(1) \rtimes \mathbb{Z}_2$ invariant

$\sigma_0^{\mathrm{p}} = -1$ and $\sigma_{1/2}^{\mathrm{p}} = 1$

$\mathscr{R} H_s^{\mathrm{p}} = H_{1-s}^{\mathrm{p}}$ where $\mathscr{R}$ is the spatial reflection

Theorem Tasaki, in preparation

In the above setting the winding number $p \in \mathbb{Z}$, which is the total spin pumped in the adiabatic process $s : 0 \to 1$, is nonzero.

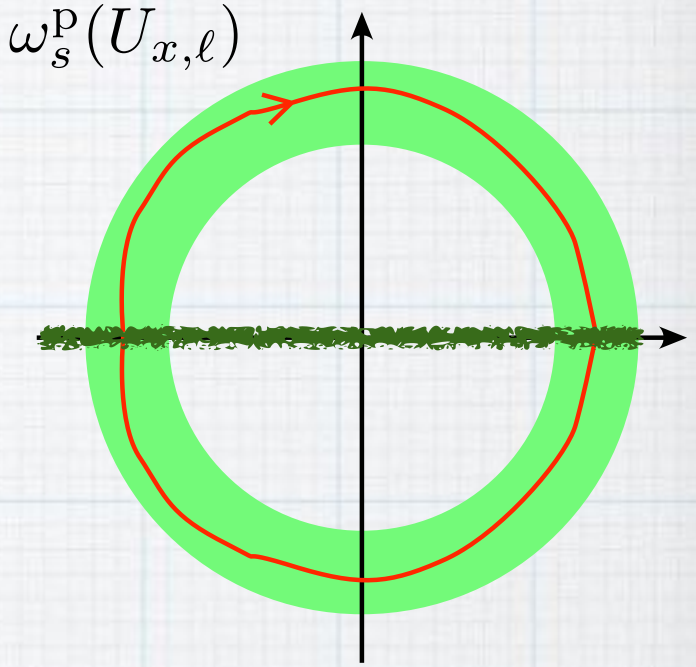


variation #7

SPT phases and spin pumping

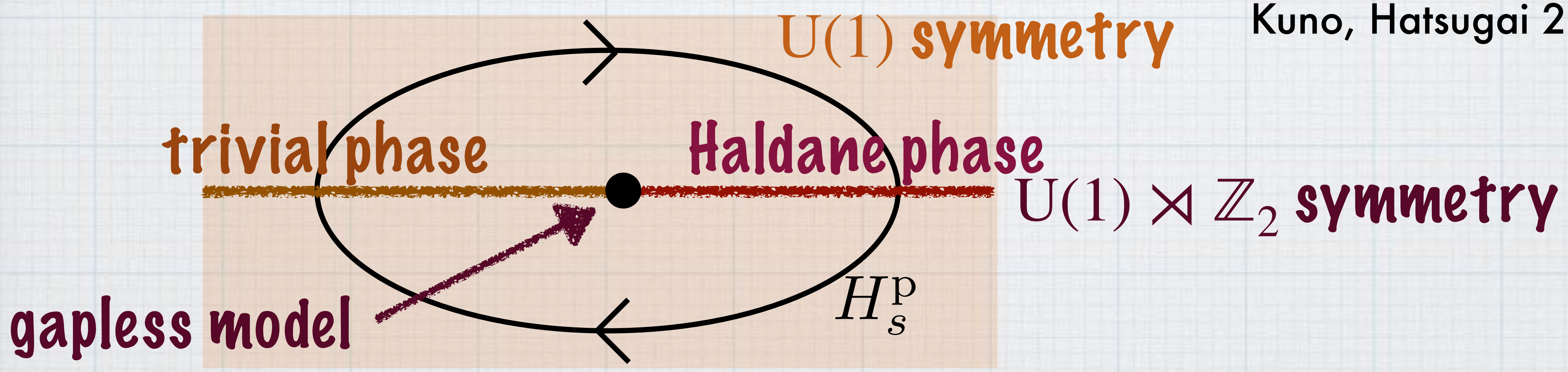
Theorem Tasaki, in preparation

In the above setting the winding number $p \in \mathbb{Z}$, which is the total spin pumped in the adiabatic process $s : 0 \to 1$, is nonzero.



we always have nonzero pumping when there is a path of Hamiltonians with a unique gapped g.s. which connect models in the Haldane phase and the trivial phase

Kuno, Hatsugai 2020



similar examples of transl. inv. models with any integer S

classifying loops of Hamiltonians

index for a loop of Hamiltonians

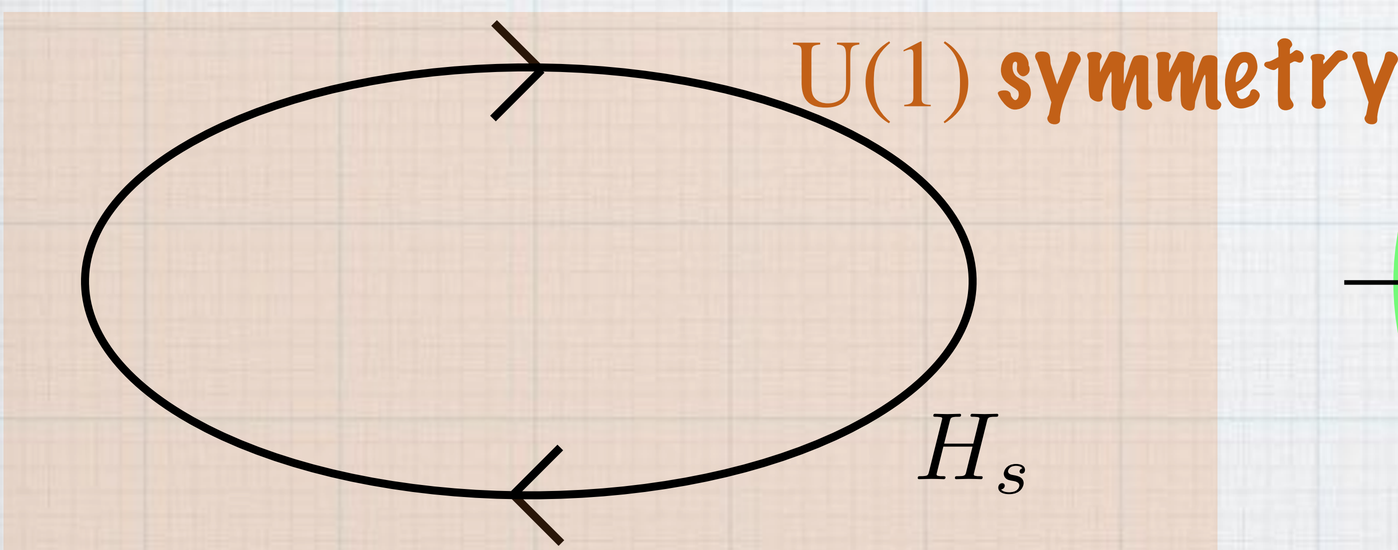
with any S, not necessarily translation invariant

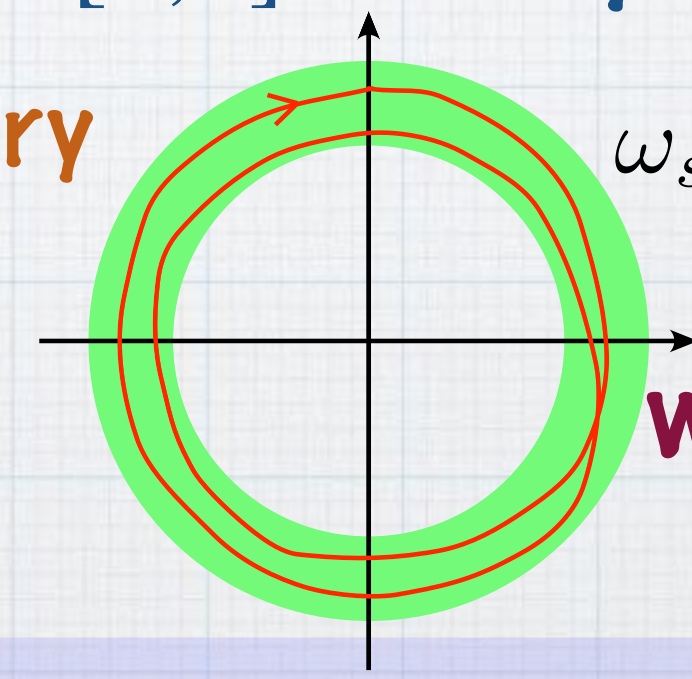
loop $H_{.}$ of Hamiltonians

one-parameter family H_s with $s \in [0,1]$ of $\mathrm{U}(1)$ invariant Hamiltonians on $\mathbb{Z}$ such that $H_0 = H_1$

H_s has a unique gapped g.s. ω_s with gap $\geq \Delta E_0 > 0$

$\omega_s(A)$ is continuous in $s \in [0,1]$ for any $A \in \mathfrak{A}_{\mathrm{loc}}$





Theorem

For each loop $H_{.}$ of Hamiltonians, there is a well-defined index $p(H_{.}) \in \mathbb{Z}$

homotopic loops of Hamiltonians

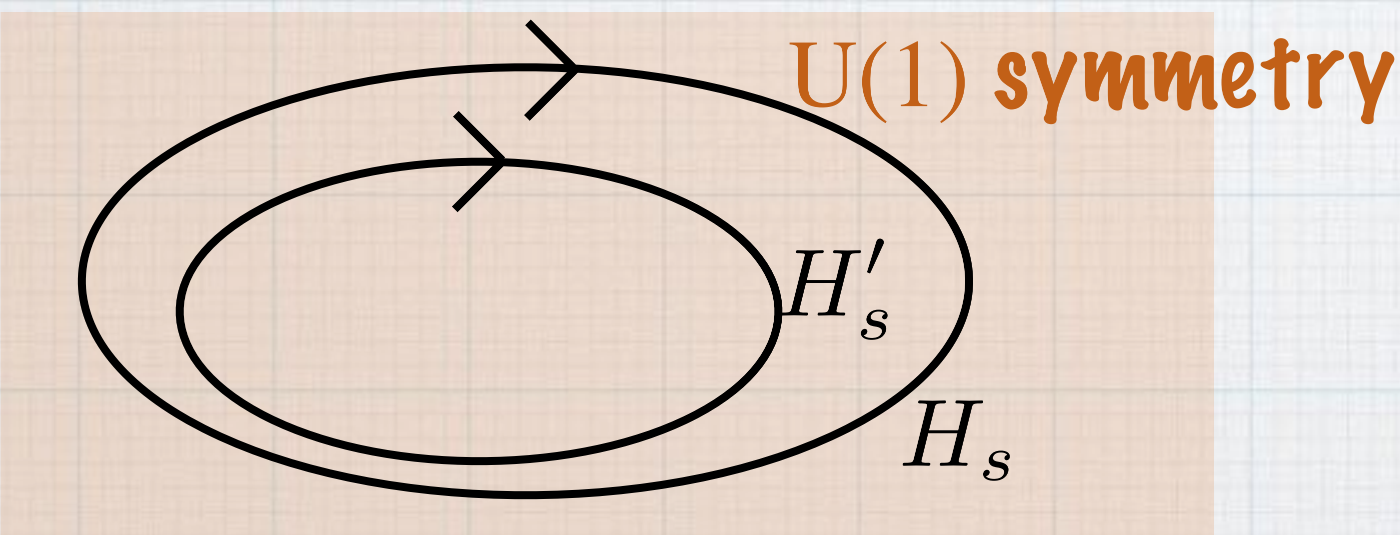
Kitaev 2013
Bachmann, DeRoeck, Fraas, Jappens 2022

loops $H_\cdot$ and $H'_\cdot$ of Hamiltonians are homotopic iff

$\exists$family $\tilde{H}_{s,\lambda}$ with $s, \lambda \in [0,1]$ of $\mathrm{U}(1)$ invariant Hamiltonians on $\mathbb{Z}$ such that $\tilde{H}_{s,0} = H_s$ and $\tilde{H}_{s,1} = H'_s$

$\tilde{H}_{s,\lambda}$ has a unique gapped g.s. $\omega_{s,\lambda}$ with gap $\geq \Delta E_1 > 0$

$\omega_{s,\lambda}(A)$ is continuous in $(s, \lambda) \in [0,1]$ for any $A \in \mathfrak{A}_{\mathrm{loc}}$

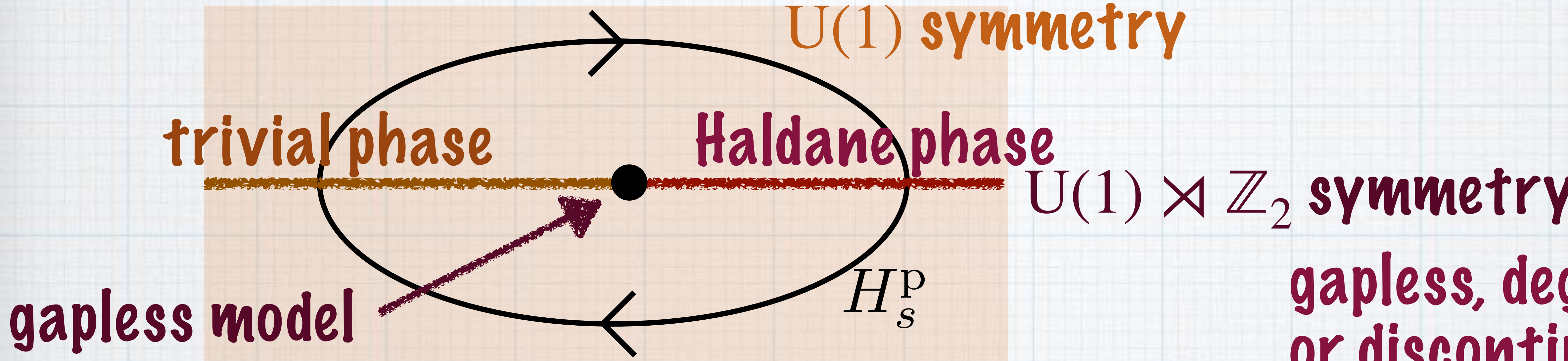


variation #8

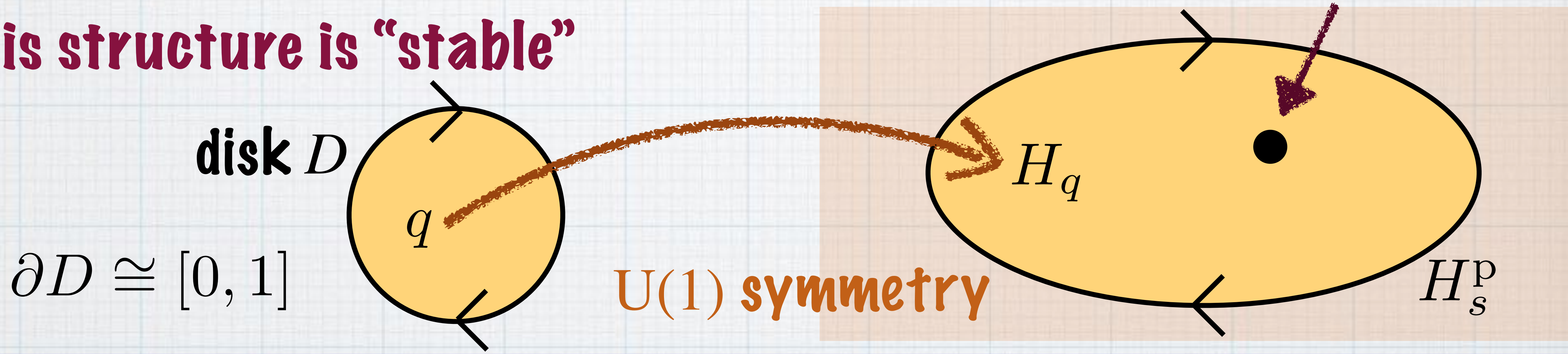
Theorem Tasaki, in preparation

If $H_\cdot$ and $H'_\cdot$ are homotopic, then $p(H_\cdot) = p(H'_\cdot)$

another no-go theorem



this structure is "stable"



Corollary

Let D be a disk, and H_q be $\mathrm{U}(1)$ invariant Hamiltonians indexed by $q \in D$ which coincide with H_s^{p} on ∂D. Then it is impossible that H_q has a unique gapped g.s. ω_q for all $q \in D$ and that $\omega_q(A)$ is continuous in q for any $A \in \mathfrak{A}_{\mathrm{loc}}$

summary

- ☑ the twist operator of Lieb-Schultz-Mattis and Affleck-Lieb enables us to prove various "topological" properties of quantum spin chains in surprisingly elementary manners.

- ☑ the methods apply (more naturally) to fermionic or bosonic systems in one dimension.

- ☑ are there similar elementary strategies that apply to systems with discrete symmetry?

references

background, basics, and other interesting topics

Hal Tasaki "Physics and Mathematics of Quantum Many-Body Systems" (Springer, 2020)

basics of the twist operator, generalized and extended Lieb-Schultz-Mattis theorems (pedagogical review)

Hal Tasaki "The Lieb-Schultz-Mattis Theorem: A Topological Point of View" in "the Lieb 90 volume" (EMS, 2022) arXiv:2202.06243

$\mathbb{Z}_2$ index and SPT phase transitions

Hal Tasaki "Topological phase transition and $\mathbb{Z}_2$ index for $S = 1$ quantum spin chains" PRL 121,140604 (2018) arXiv:1804.04337

Hal Tasaki "Rigorous Index Theory for One-Dimensional Interacting Topological Insulators" arXiv:2111.07335

the pumping state that connects the trivial and the AKLT model

Daisuke Maekawa and Hal Tasaki "The Asymmetric Valence-Bond-Solid States in Quantum Spin Chains: The Difference Between Odd and Even Spins" arXiv:2205.00653