ENTROPY AND "THERMODYNAMIC" RELATIONS FOR NONEQUILIBRIUM STEADY STATES

HAL TASAKI
WITH T.S.KOMATSU, N.NAKAGAWA, S.SASA

PRL 100, 230602 (2008), arXiv:0711.0246
J. STAT. PHYS. 159, 1237 (2015), arXiv:1405.0697

webinar, November 2020

TWO "TWISTS" IN STEADY STATE THERMODYNAMICS

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MOTIVATION

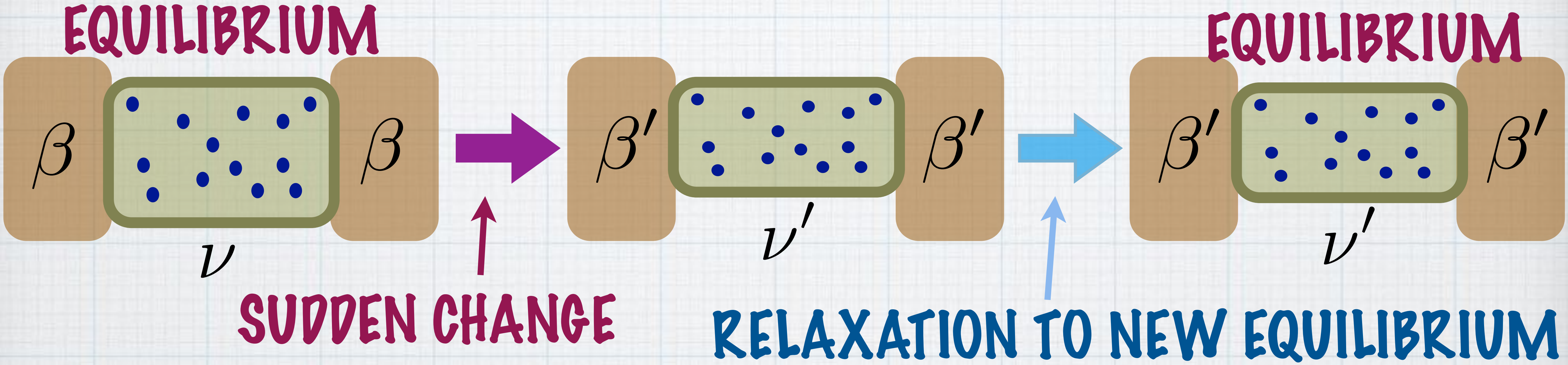
EQUILIBRIUM THERMODYNAMICS

PHYSICAL SYSTEM WITH CONTROLLABLE PARAMETERS ν

E.G., THE VOLUME

START FROM THE EQUILIBRIUM WITH $\beta = T^{-1}, \nu$

CHANGE THE PARAMETERS TO β', ν'



ΔQ ENERGY (HEAT) TRANSFERRED TO THE BATHS FROM THE SYSTEM DURING THE RELAXATION PROCESS

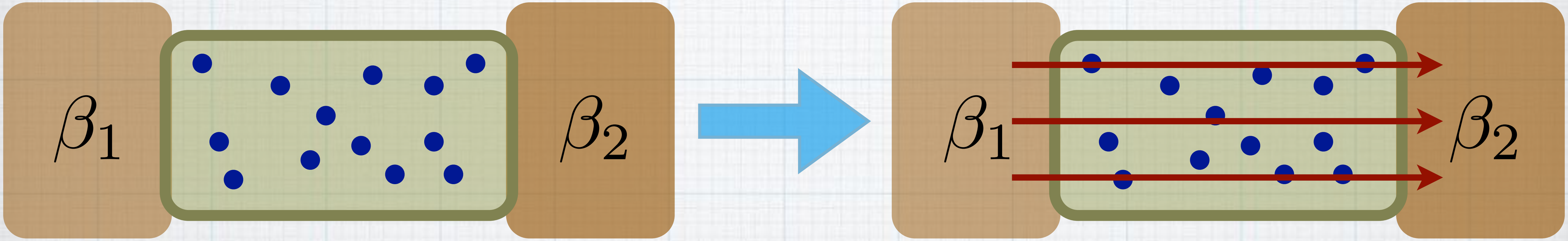
CLAUSIUS RELATION

$$S(\beta', \nu') - S(\beta, \nu) = -\beta\, \Delta Q + O((\Delta Q)^2)$$

STARTING POINT OF EQUILIBRIUM THERMODYNAMICS

NONEQUILIBRIUM STEADY STATE (NESS)

SET $\beta_1 \neq \beta_2$, AND FIX β_1, β_2, ν



AS $t \uparrow \infty$ THE SYSTEM IS EXPECTED TO APPROACH A UNIQUE STATIONARY STATE = NONEQUILIBRIUM STEADY STATE (NESS)

(PROVIDED THAT THE "DEGREE OF NONEQULIBRIUM" IS SMALL)

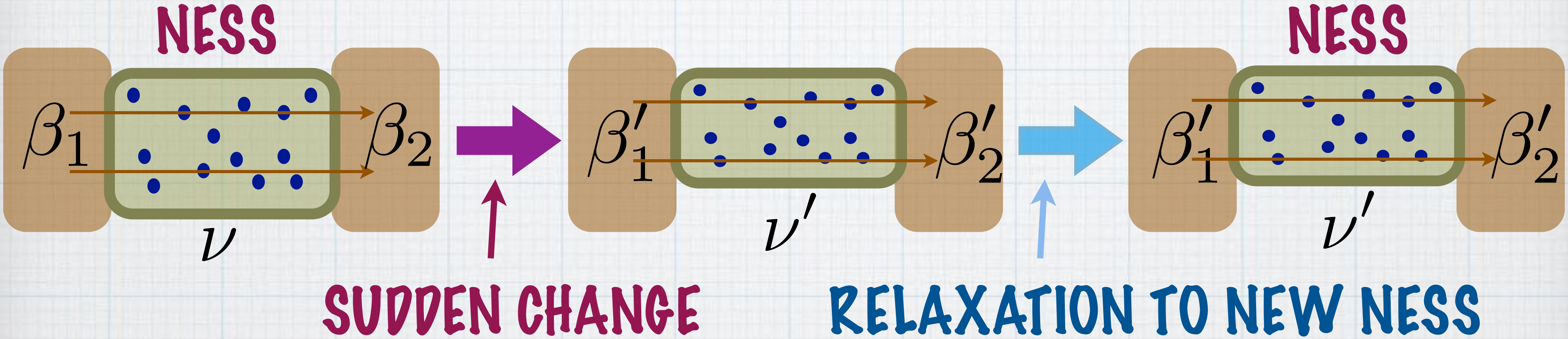
EQUILIBRIUM STATE:
- ☑ NO MACROSCOPIC CHANGES
- ☑ NO MACROSCOPIC FLOWS

NESS:
- ☑ NO MACROSCOPIC CHANGES
- ☑ NONVANISHING MACROSCOPIC FLOW OF ENERGY OR MATTER

OPERATION TO NESS
THERMODYNAMICS FOR NESS?

START FROM THE NESS WITH β_1, β_2, ν

CHANGE THE PARAMETERS TO β'_1, β'_2, ν'



IS THERE ANYTHING LIKE THE CLAUSIUS RELATION?

DERIVATION BASED ON GENERAL MICROSCOPIC MODELS

TYPICAL SYSTEM

SYSTEM OF PARTICLES

CLASSICAL MECHANICAL SYSTEM WITH N PARTICLES IN A FINITE BOX Λ

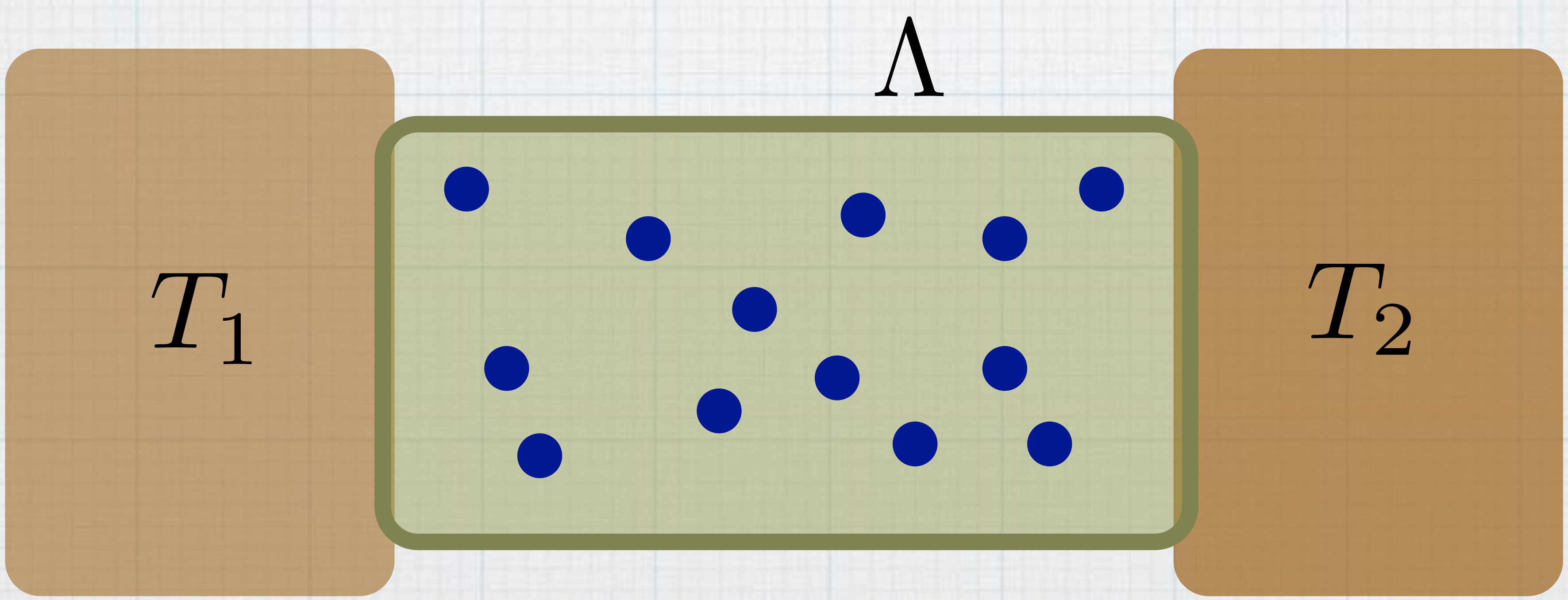
$\boldsymbol{r}_i \in \Lambda \subset \mathbb{R}^3$ POSITION

OF THE i-TH PARTICLE

$\boldsymbol{p}_i \in \mathbb{R}^3$ MOMENTUM

$$\boldsymbol{p}_i = m\boldsymbol{v}_i$$

$$x = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$$



THE SYSTEM IS ATTACHED TO TWO HEAT BATHS

TIME EVOLUTION

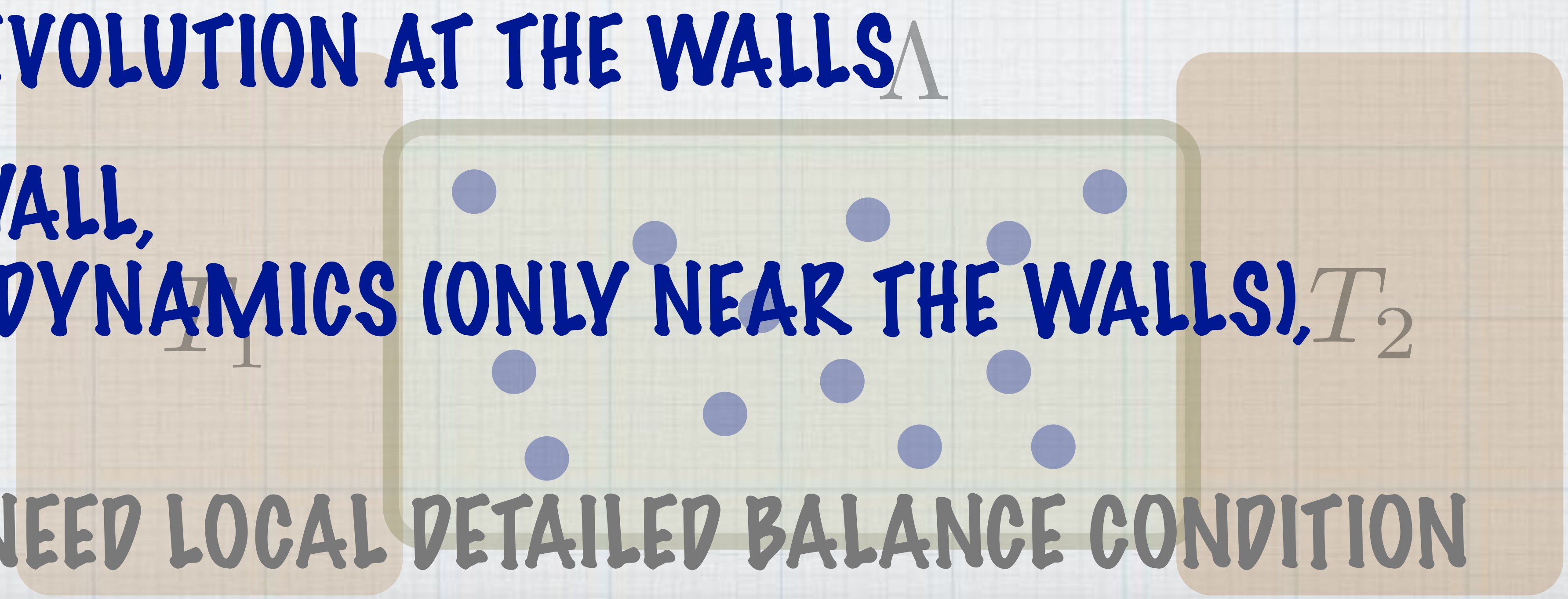
USUAL NEWTON EQUATION

$$m\frac{d^2\boldsymbol{r}_i(t)}{dt^2} = -\mathrm{grad}_i V(\boldsymbol{r}_1(t), \ldots, \boldsymbol{r}_N(t))$$

$$V(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \sum_i V_{\mathrm{ext}}^{(\nu)}(\boldsymbol{r}_i) + \sum_{i<j} v(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

ν (CONTROLLABLE) PARAMETER (E.G., THE VOLUME)

MARKOVIAN TIME EVOLUTION AT THE WALLS



- ☑ THERMAL WALL,
- ☑ LANGEVIN DYNAMICS (ONLY NEAR THE WALLS),
- ☑ ETC.

WE NEED LOCAL DETAILED BALANCE CONDITION

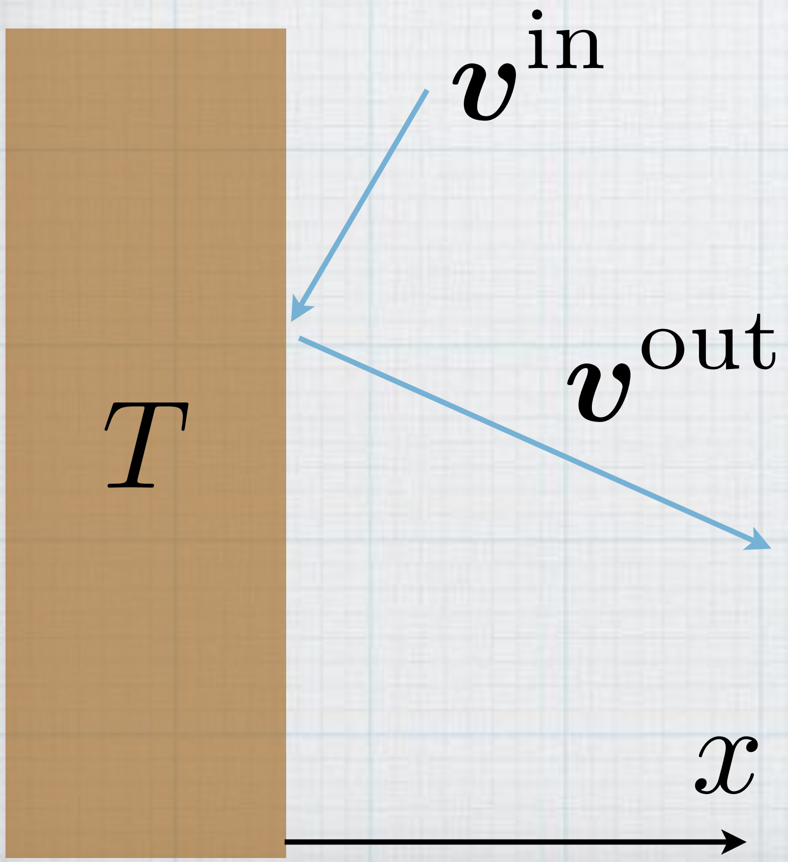
THERMAL WALL

A PARTICLE WITH ANY INCIDENT VELOCITY $\boldsymbol{v}^{\mathrm{in}}$ IS BOUNCED BACK WITH A RANDOM VELOCITY $\boldsymbol{v}^{\mathrm{out}}$ WITH THE PROBABILITY DENSITY

$$p_T(\boldsymbol{v}^{\mathrm{out}}) = A\, v_{\mathrm{x}}^{\mathrm{out}} \exp\Big[-\frac{m\,|\boldsymbol{v}^{\mathrm{out}}|^2}{2kT}\Big]$$

$$A = \frac{1}{2\pi}\Big(\frac{m}{kT}\Big)^2$$

$$\boldsymbol{v}^{\mathrm{out}} = (v_{\mathrm{x}}^{\mathrm{out}}, v_{\mathrm{y}}^{\mathrm{out}}, v_{\mathrm{z}}^{\mathrm{out}}) \qquad v_{\mathrm{x}}^{\mathrm{out}} > 0 \quad v_{\mathrm{y}}^{\mathrm{out}}, v_{\mathrm{z}}^{\mathrm{out}} \in \mathbb{R}$$



k BOLTZMANN CONSTANT

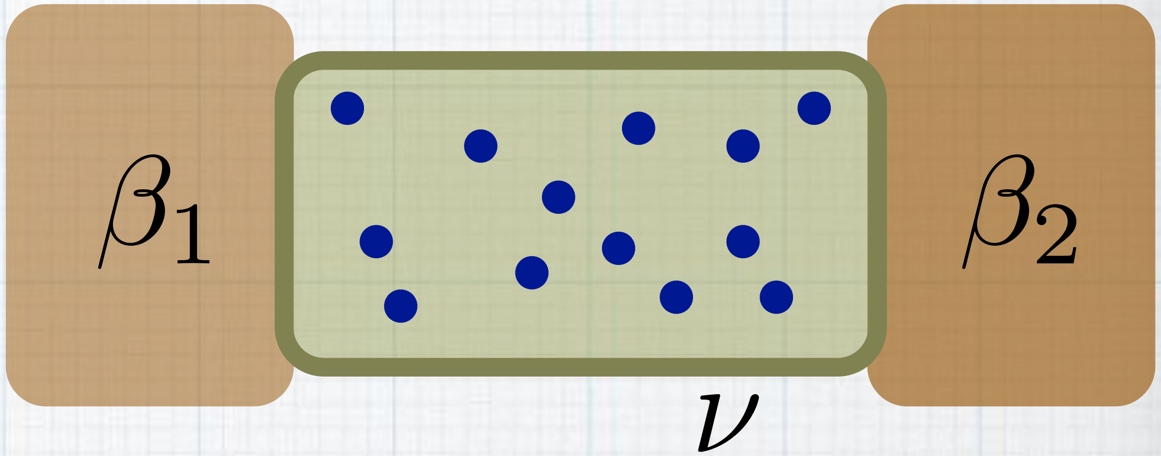
T TEMPERATURE OF THE WALL

ENERGY (HEAT) TRANSFERED FROM THE SYSTEM TO THE BATH

$$q = \frac{m}{2}|\boldsymbol{v}^{\mathrm{in}}|^2 - \frac{m}{2}|\boldsymbol{v}^{\mathrm{out}}|^2$$

GENERAL SETUP

BASIC INGREDIENTS

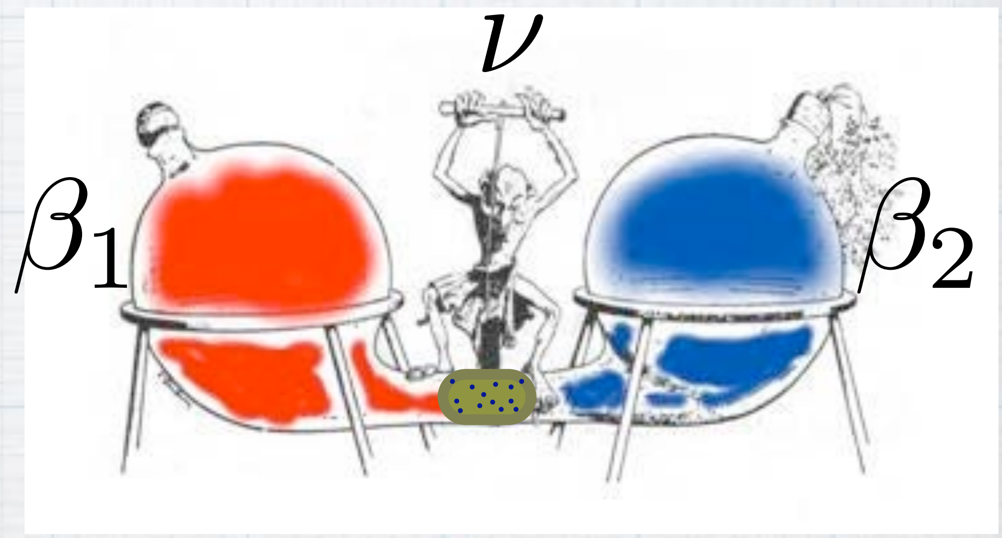


CONTROLLABLE PARAMETERS

$$\alpha = (\beta_1, \beta_2, \nu)$$

(INVERSE) TEMPERATURES OF THE BATHS

MODEL PARAMETERS



x STATE OF THE SYSTEM

$$x \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$$

$$x^* \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; -\boldsymbol{p}_1, \ldots, -\boldsymbol{p}_N)$$

ENERGY $H_x^\nu = H_{x^*}^\nu$

TIME EVOLUTION

TIME INTERVAL $t \in [0, \tau]$

OPERATION BY OUTSIDE AGENT

FIXED PROTOCOL (OR FUNCTION)

$$\alpha(t) = (\beta_1(t), \beta_2(t), \nu(t)) \qquad \hat{\alpha} = (\alpha(t))_{t\in[0,\tau]}$$

PATH $\hat{x} = (x(t))_{t\in[0,\tau]}$

MARKOV DYNAMICS WITH PATH PROBABILITY DENSITY $\mathcal{T}^{\hat{\alpha}}[\hat{x}]$

$$\int_{x(0)=x_{\mathrm{init}}} D\hat{x}\, \mathcal{T}^{\hat{\alpha}}[\hat{x}] = 1$$

DETAILED FLUCTUATION THEOREM

$$\mathcal{T}^{\hat{\alpha}^\dagger}[\hat{x}^\dagger] = e^{-\Theta^{\hat{\alpha}}[\hat{x}]}\, \mathcal{T}^{\hat{\alpha}}[\hat{x}]$$

TIME REVERSED PROTOCOL $\hat{\alpha}^\dagger = (\alpha(\tau - t))_{t\in[0,\tau]}$

TIME REVERSED PATH $\hat{x}^\dagger = (x^*(\tau - t))_{t\in[0,\tau]}$

ENTROPY PRODUCTION

DETAILED FLUCTUATION THEOREM

$$\mathcal{T}^{\hat{\alpha}^\dagger}[\hat{x}^\dagger] = e^{-\Theta^{\hat{\alpha}}[\hat{x}]}\,\mathcal{T}^{\hat{\alpha}}[\hat{x}]$$

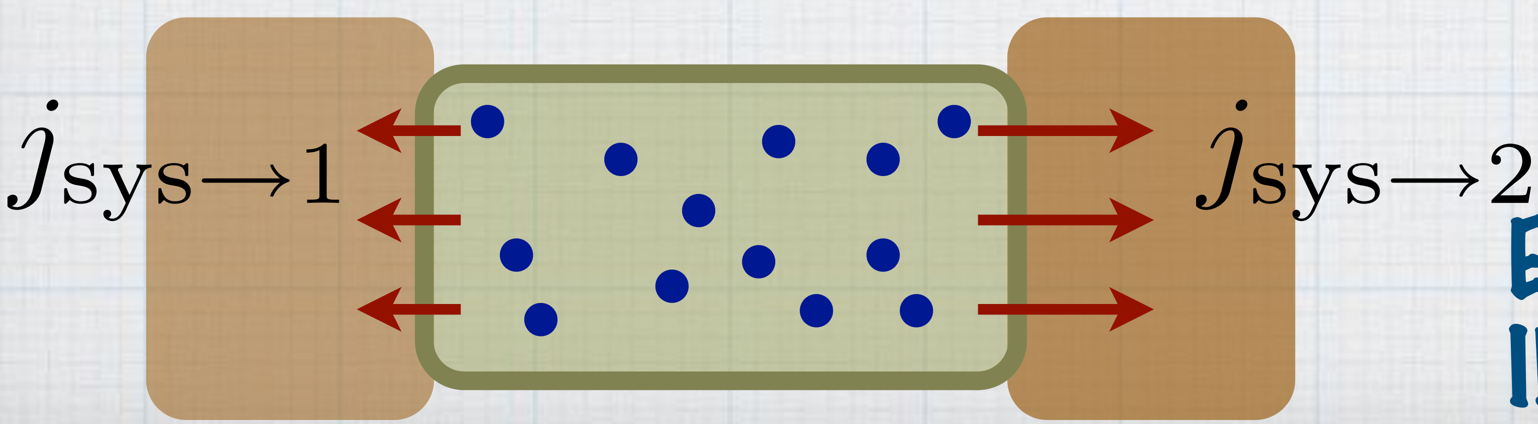
$\Theta^{\hat{\alpha}}[\hat{x}]$ TOTAL ENTROPY PRODUCTION (IN THE BATHS)

WHEN THE TEMPERATURES ARE KEPT CONSTANT

$$\Theta^{\hat{\alpha}}[\hat{x}] = \beta_1 Q_{\mathrm{sys}\to 1}[\hat{x}] + \beta_2 Q_{\mathrm{sys}\to 2}[\hat{x}]$$

IN GENERAL

$$\Theta^{\hat{\alpha}}[\hat{x}] = \int_0^{\tau} dt \sum_{i=1,2} \beta_i(t)\, j_{\mathrm{sys}\to i}[\hat{x}](t)$$



ENTROPY PRODUCTION RATE
IN THE i-TH BATH

NESS AND THE AVERAGE

WHEN THE PARAMETERS ARE KEPT CONSTANT
THE SYSTEM EVENTUALLY CONVERGES TO A UNIQUE
NONEQUILIBRIUM STEADY STATE (NESS)

ρ_x^α PROBABILITY DISTRIBUTION OF NESS WITH α

PATH AVERAGE OF ANY FUNCTION $F[\hat{x}]$

$$\langle F \rangle^{\hat{\alpha}} := \int \mathcal{D}\hat{x}\, \rho_{x(0)}^{\alpha(0)}\, \mathcal{T}^{\hat{\alpha}}[\hat{x}]\, F[\hat{x}]$$

START FROM THE NESS FOR THE INITIAL PARAMETERS
AND CHANGE THE PARAMETER ACCORDING TO $\hat{\alpha}$

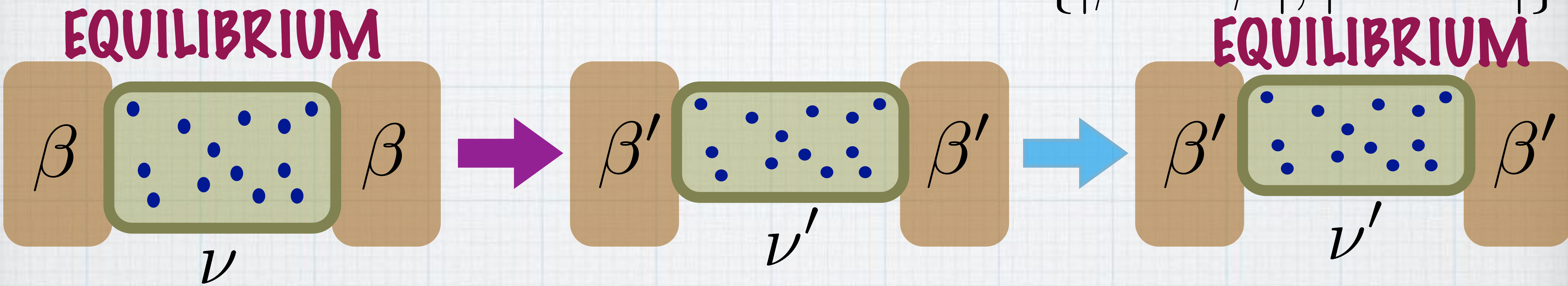
CLAUSIUS RELATION AND EXTENDED CLAUSIUS RELATION

EQUILIBRIUM CASE

OPERATION BETWEEN TWO EQUILIBRIUM STATES

$$\alpha(t) = \begin{cases} (\beta, \beta, \nu), & t \in [0, \tau/2] \\ (\beta', \beta', \nu'), & t \in (\tau/2, \tau] \end{cases}$$

AMOUNT OF CHANGE $\delta = \max\{|\beta' - \beta|, |\nu' - \nu|\}$



STANDARD CLAUSIUS RELATION (FOR LARGE τ)

$$S(\beta', \beta', \nu') - S(\beta, \beta, \nu) = -\langle \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}} + O(\delta^2)$$

THERMODYNAMIC ENTROPY = SHANNON ENTROPY OF ρ^α

$$S(\alpha) = -\int dx \rho_x^\alpha \log \rho_x^\alpha$$

THE MEANING OF THE CLAUSIUS RELATION

STANDARD CLAUSIUS RELATION

$$S(\beta',\beta',\nu') - S(\beta,\beta,\nu) = -\langle\Theta^{\hat{\alpha}}\rangle^{\hat{\alpha}} + O(\delta^2)$$

$$\langle\Theta^{\hat{\alpha}}\rangle^{\hat{\alpha}} \to S^{\text{fin}}_{\text{baths}} - S^{\text{init}}_{\text{baths}}$$

$$S(\beta,\beta,\nu) + S^{\text{init}}_{\text{baths}} = S(\beta',\beta',\nu') + S^{\text{fin}}_{\text{baths}} + O(\delta^2)$$

THE TOTAL ENTROPY OF {THE SYSTEM + THE BATHS} IS CONSTANT

THIS IS NO LONGER TRUE IN NESS!

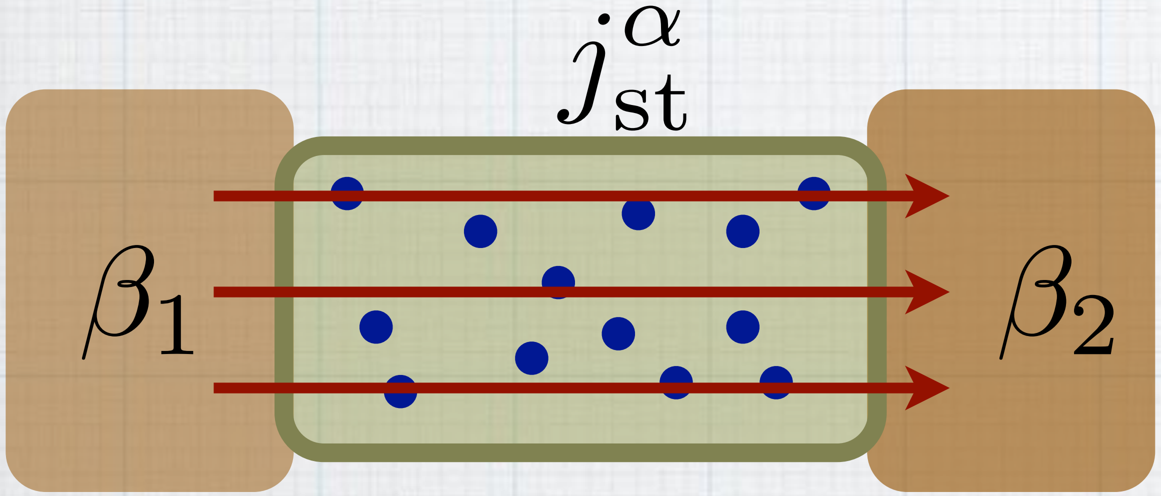
THERE IS A CONSTANT ENTROPY PRODUCTION

ENTROPY PRODUCTION IN NESS

ENTROPY PRODUCTION RATE IN NESS

$$\sigma_{\rm st}^{\alpha} = \beta_2 j_{\rm st}^{\alpha} - \beta_1 j_{\rm st}^{\alpha} \propto (\beta_2 - \beta_1)^2$$

STATIONARY CURRENT $j_{\rm st}^{\alpha} \propto \beta_2 - \beta_1$



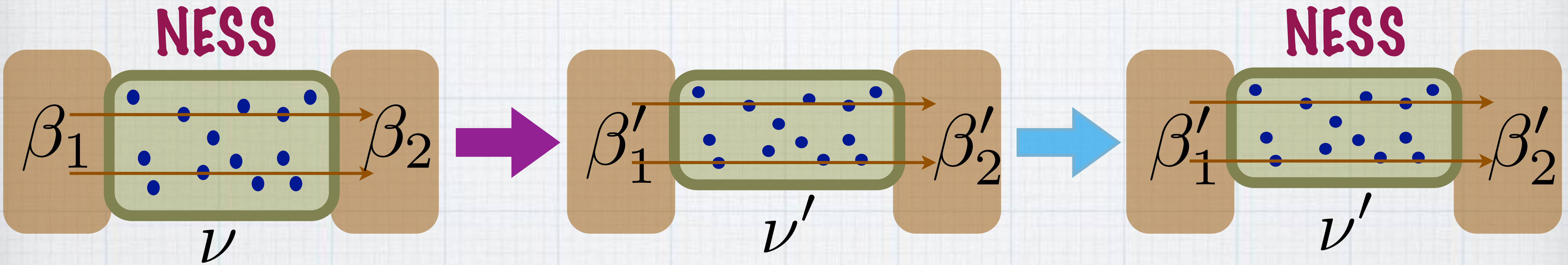
$\sigma_{\rm st}^{\alpha} = 0$	$\beta_1 = \beta_2$
$\sigma_{\rm st}^{\alpha} > 0$	$\beta_1 \neq \beta_2$

NESS IS ACCOMPANIED BY A CONSTANT NONVANISHING ENTROPY PRODUCTION IN THE BATHS

CLAUSIUS RELATION FOR NESS?

OPERATION BETWEEN TWO NESS

$$\alpha(t) = \begin{cases} (\beta_1, \beta_2, \nu), & t \in [0, \tau/2] \\ (\beta_1', \beta_2', \nu'), & t \in (\tau/2, \tau] \end{cases}$$



IS IT POSSIBLE THAT $S(\beta_1', \beta_2', \nu') - S(\beta_1, \beta_2, \nu) \simeq -\langle \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}}$?

NO! BECAUSE

$S(\beta_1', \beta_2', \nu') - S(\beta_1, \beta_2, \nu)$ IS INDEPENDENT OF τ

$\langle \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}} \sim \frac{\tau}{2}\,\sigma_{\rm st}^{\alpha} + \frac{\tau}{2}\,\sigma_{\rm st}^{\alpha'}$ DIVERGES AS $\tau \uparrow \infty$

ENTROPY PRODUCTION

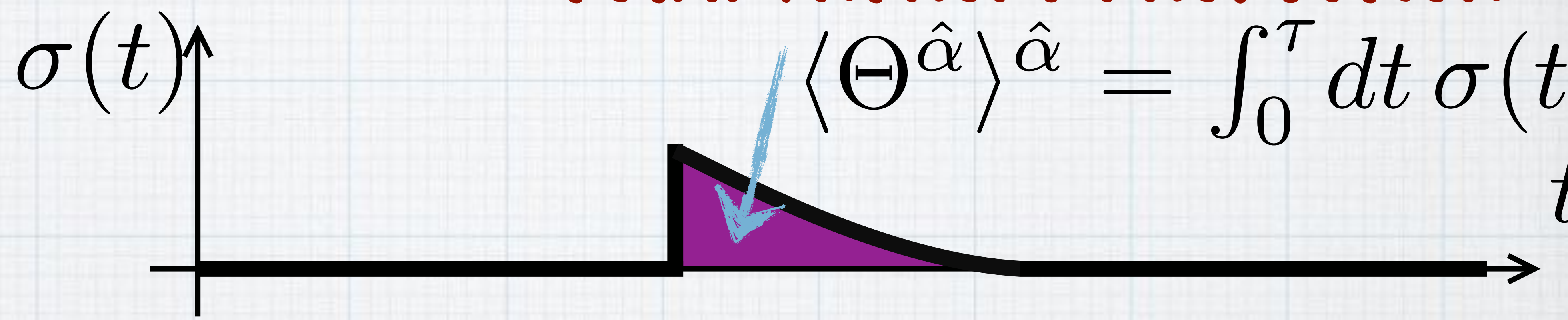
ENTROPY PRODUCTION RATE

$$\sigma(t) = \beta_1\, j_{\mathrm{sys}\to 1}(t) + \beta_2\, j_{\mathrm{sys}\to 2}(t)$$

TOTAL ENTROPY PRODUCTION

$$\langle \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}} = \int_0^\tau dt\, \sigma(t)$$

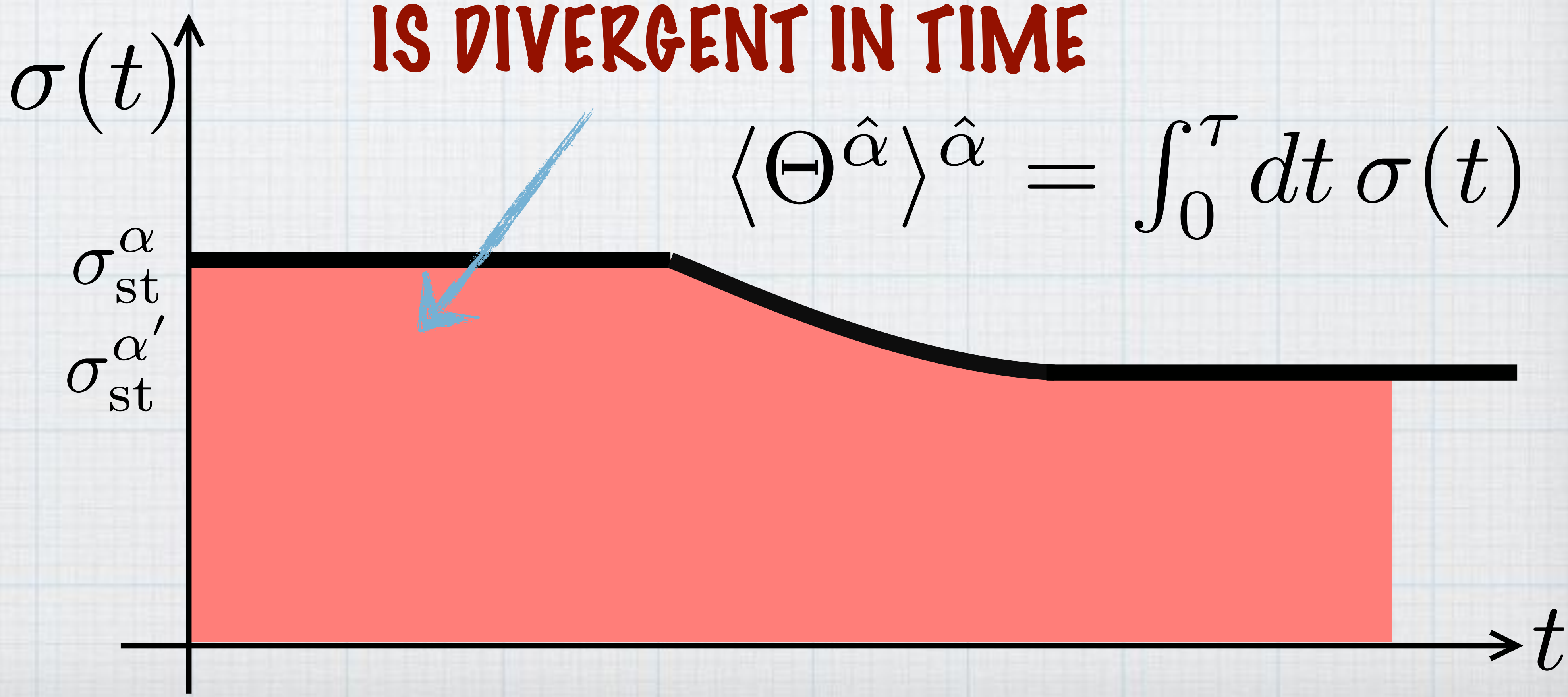
OPERATION IN EQUILIBRIUM



TOTAL ENTROPY PRODUCTION IS DIVERGENT IN TIME

$$\langle \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}} = \int_0^\tau dt\, \sigma(t)$$

OPERATION IN NESS



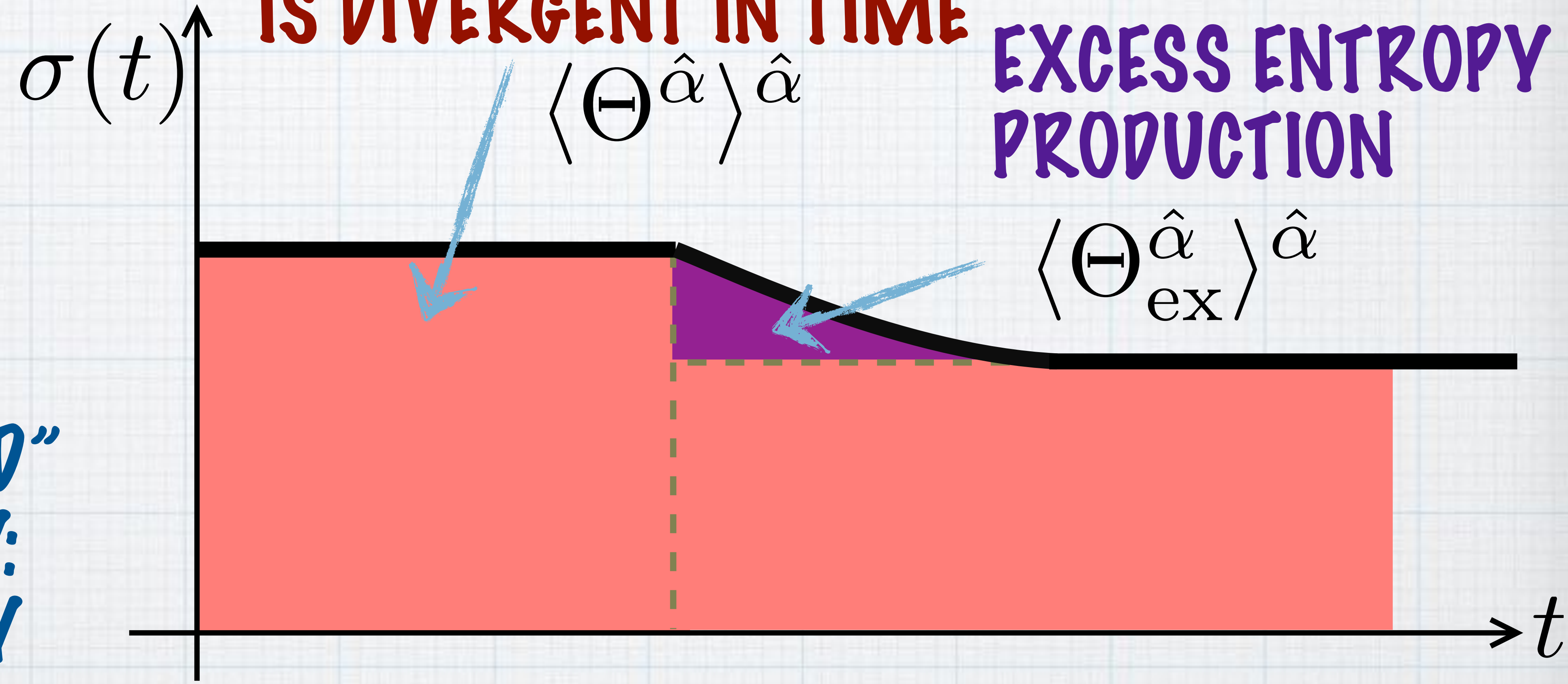
EXCESS ENTROPY PRODUCTION

OONO-PANICONI

OPERATION IN NESS

TOTAL ENTROPY PRODUCTION IS DIVERGENT IN TIME

EXCESS ENTROPY PRODUCTION



“RENORMALIZED” FINITE QUANTITY: EXCESS ENTROPY PRODUCTION

$$\Theta^{\hat{\alpha}}_{\text{ex}}[\hat{x}] := \Theta^{\hat{\alpha}}[\hat{x}] - \int_0^{\tau} dt\, \sigma^{\alpha(t)}_{\text{st}}$$

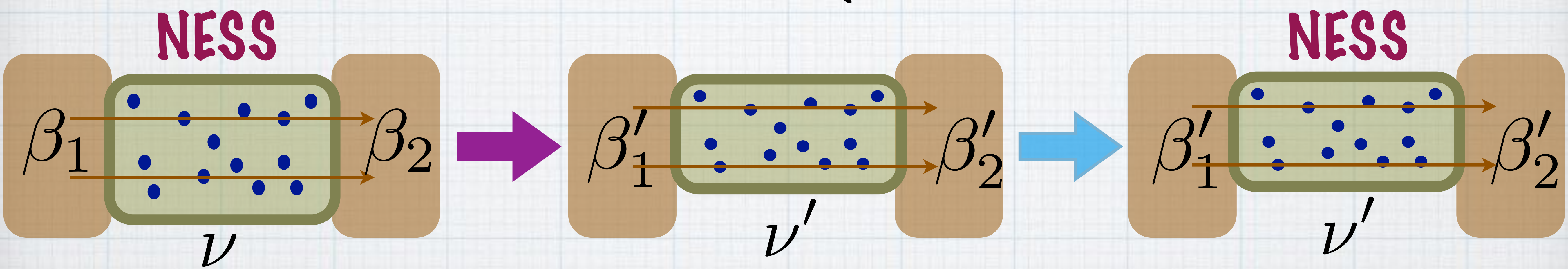
“BARE” ENTROPY PRODUCTION

“HOUSE KEEPING” ENTROPY PRODUCTION

EXTENDED CLAUSIUS RELATION

OPERATION BETWEEN TWO NESS

$$\alpha(t) = \begin{cases} (\beta_1, \beta_2, \nu), & t \in [0, \tau/2] \\ (\beta_1', \beta_2', \nu'), & t \in (\tau/2, \tau] \end{cases}$$



THERE EXISTS ENTROPY OF NESS, AND WE HAVE

$$S(\beta_1', \beta_2', \nu') - S(\beta_1, \beta_2, \nu) = -\langle \Theta_{\mathrm{ex}}^{\hat{\alpha}} \rangle^{\hat{\alpha}} + O(\epsilon^2 \delta) + O(\delta^2)$$

AMOUNT OF CHANGE $\delta = \max\{|\beta_1' - \beta_1|, |\beta_2' - \beta_2|, |\nu' - \nu|\}$

DEGREE OF NONEQUILIBRIUM $\epsilon = \max\{|\beta_1 - \beta_2|, |\beta_1' - \beta_2'|\}$

MICROSCOPIC EXPRESSION FOR THE ENTROPY

THE NONEQUILIBRIUM ENTROPY IS RELATED TO THE PROBABILITY DENSITY BY

$$S(\alpha) = S_{\mathrm{sym}}[\rho^{\alpha}]$$

WITH THE "SYMMETRIZED SHANNON ENTROPY"

$$S_{\mathrm{sym}}[\rho] := -\int dx\, \rho_x \, \log \sqrt{\rho_x \, \rho_{x^*}}$$

STATE $x \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$

TIME REVERSAL $x^* \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; -\boldsymbol{p}_1, \ldots, -\boldsymbol{p}_N)$

THE FIRST "TWIST" IN SST

THE NONEQUILIBRIUM ENTROPY IS RELATED TO THE PROBABILITY DENSITY BY

$$S(\alpha) = S_{\rm sym}[\rho^\alpha]$$

WITH THE "SYMMETRIZED SHANNON ENTROPY"

$$S_{\rm sym}[\rho] := -\int dx\, \rho_x \, \log \sqrt{\rho_x\, \rho_{x^*}}$$

STATE $x \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$

TIME REVERSAL $x^* \leftrightarrow (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; -\boldsymbol{p}_1, \ldots, -\boldsymbol{p}_N)$

ADIABATIC LIMIT

FOR "SLOW AND GENTLE" PROTOCOL $\hat{\alpha} = (\alpha(t))_{t\in[0,\tau]}$

WITH $\alpha(0) = (\beta_1, \beta_2, \nu)$ AND $\alpha(\tau) = (\beta_1', \beta_2', \nu')$

$$S(\beta_1', \beta_2', \nu') - S(\beta_1, \beta_2, \nu) = -\langle \Theta_{\rm ex}^{\hat{\alpha}} \rangle^{\hat{\alpha}} + O(\epsilon^2 \delta)$$

AMOUNT OF CHANGE $\delta = \max\{|\beta_1' - \beta_1|, |\beta_2' - \beta_2|, |\nu' - \nu|$

DEGREE OF NONEQUILIBRIUM $\epsilon = \max\{|\beta_1 - \beta_2|, |\beta_1' - \beta_2'|$

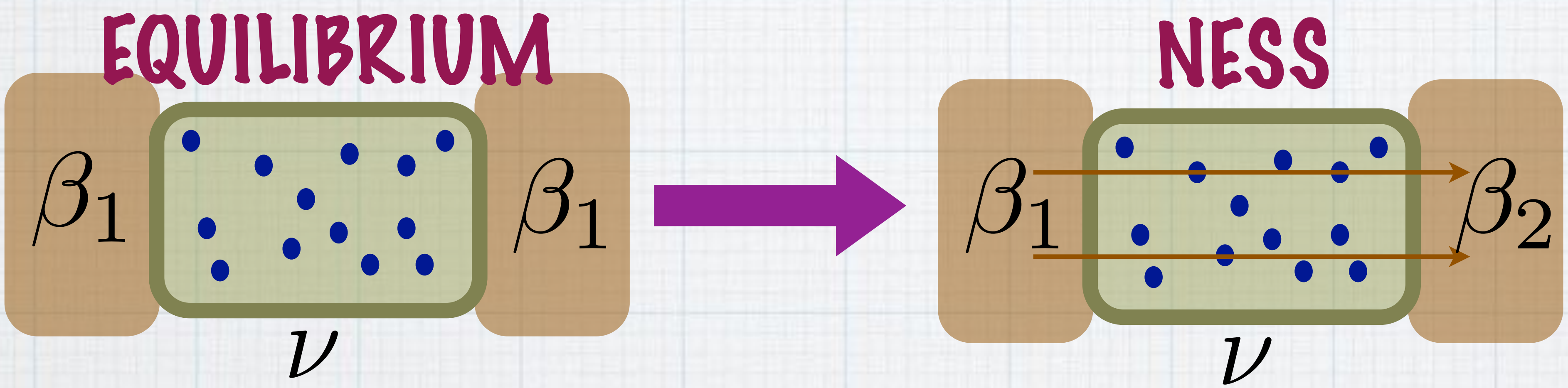
A "NATURAL" EXTENSION OF THE TRADITIONAL CLAUSIUS RELATION, IN WHICH (DIVERGENT) "BARE" ENTROPY PRODUCTION IS REPLACED BY ITS "RENORMALIZED" COUNTERPART

MEANINGFUL WHEN THE DEGREE OF NONEQULIBRIUM IS SMALL

OPERATIONAL DETERMINATION OF ENTROPY

OPERATION BETWEEN EQULIBRIUM AND NESS

$\hat{\alpha}$ PROTOCOL WHICH BRINGS (β_1, β_1, ν) TO (β_1, β_2, ν) BY CHANGING ONLY THE TEMPERATURE OF THE BATH 2.



EQUILIBRIUM ENTROPY

$$\delta = \beta_2 - \beta_1 = \epsilon$$

$$S(\beta_1, \beta_2, \nu) - S(\beta_1, \beta_1, \nu) = -\langle \Theta_{\mathrm{ex}}^{\hat{\alpha}} \rangle^{\hat{\alpha}} + O(\epsilon^2 \delta)$$

$O(\epsilon^3)$

WE CAN DETERMINE THE NONEQUILIBRIUM ENTROPY TO $O(\epsilon^2)$ ONLY BY MEASURING THE HEAT CURRENTS!

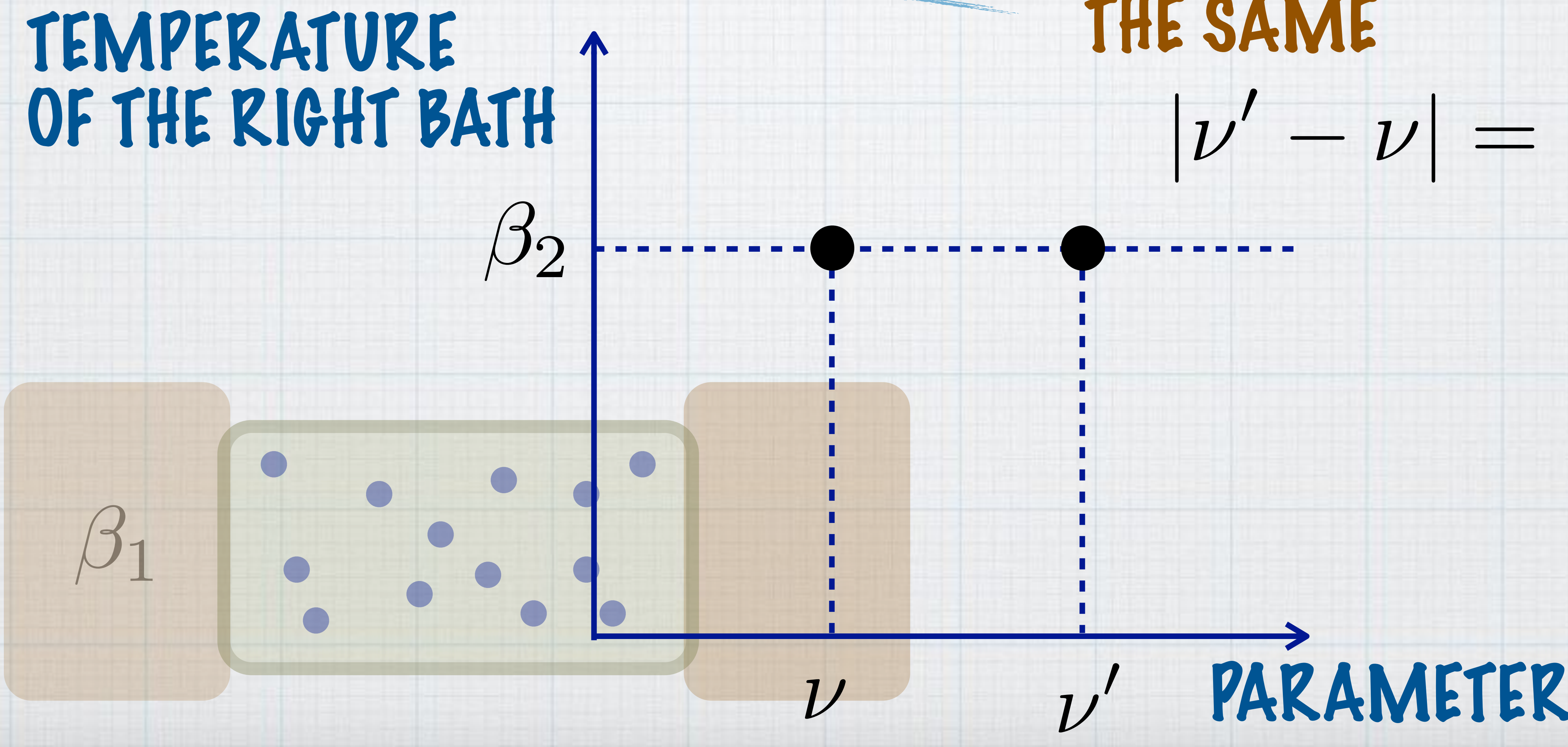
COMPARING ENTROPIES OF TWO NESS

SUPPOSE THAT WE WANT TO DETERMINE THE DIFFERENCE

$$S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu)$$

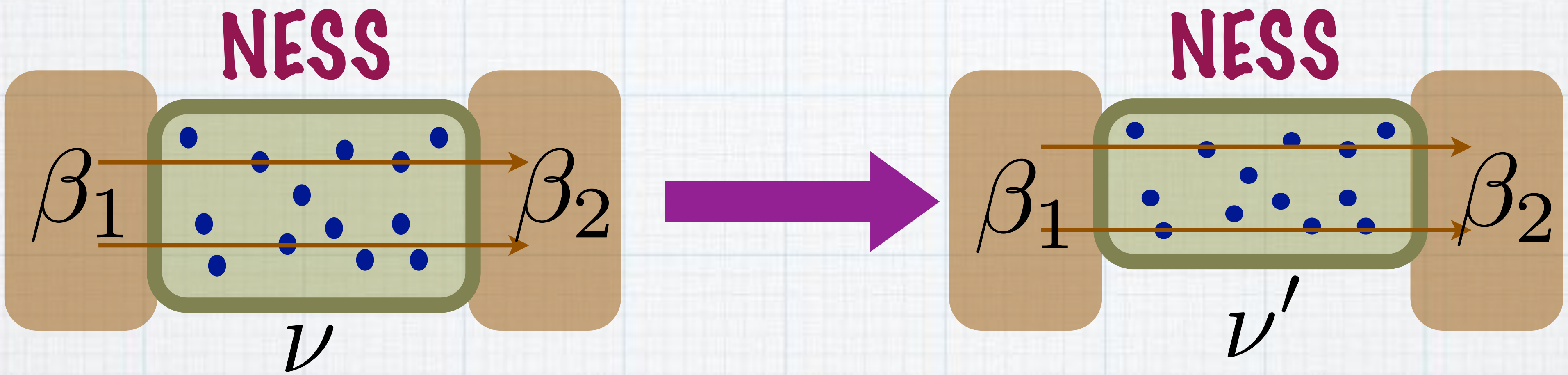
TEMPERATURES ARE THE SAME

$$|\nu' - \nu| = O(1)$$



DIRECT PATH

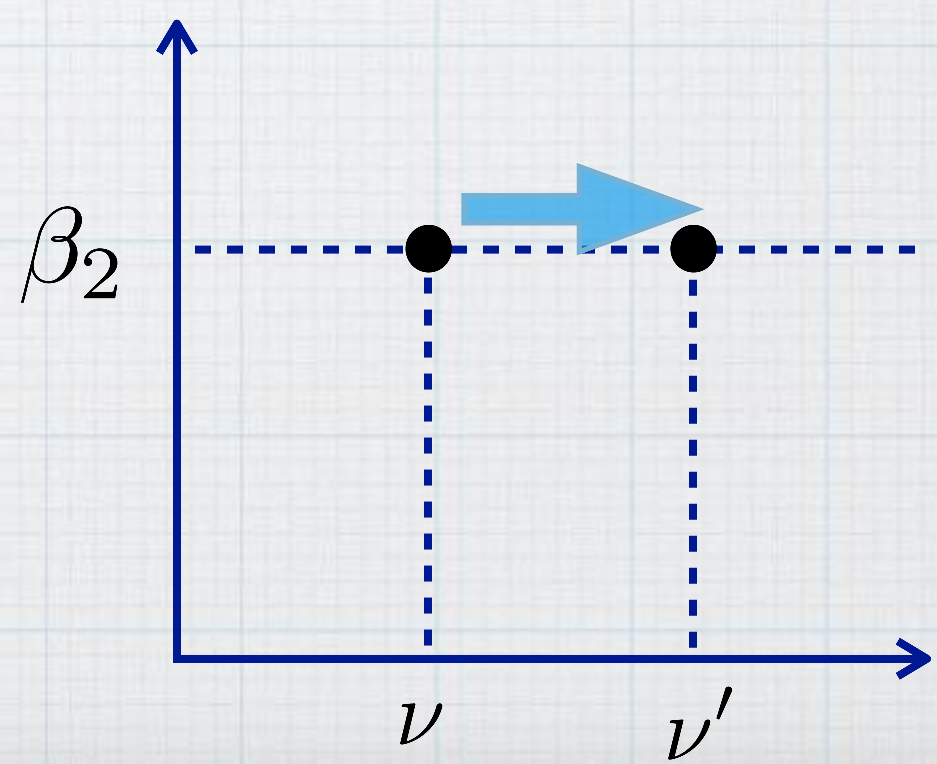
FIX THE TEMPERATURES AND CHANGE ν TO ν'



FROM THE EXTENDED CLAUSIUS RELATION, ONE GETS

$$S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu) = -\langle \Theta_{\mathrm{ex}}^{\hat{\alpha}} \rangle^{\hat{\alpha}} + \overset{O(\epsilon^2)}{\cancel{O(\epsilon^2 \delta)}}$$

$$\delta = |\nu' - \nu| = O(1)$$



WE CAN DETERMINE THE DIFFERENCE ONLY WITH THE PRECISION OF $O(\epsilon)$

INDIRECT PATH

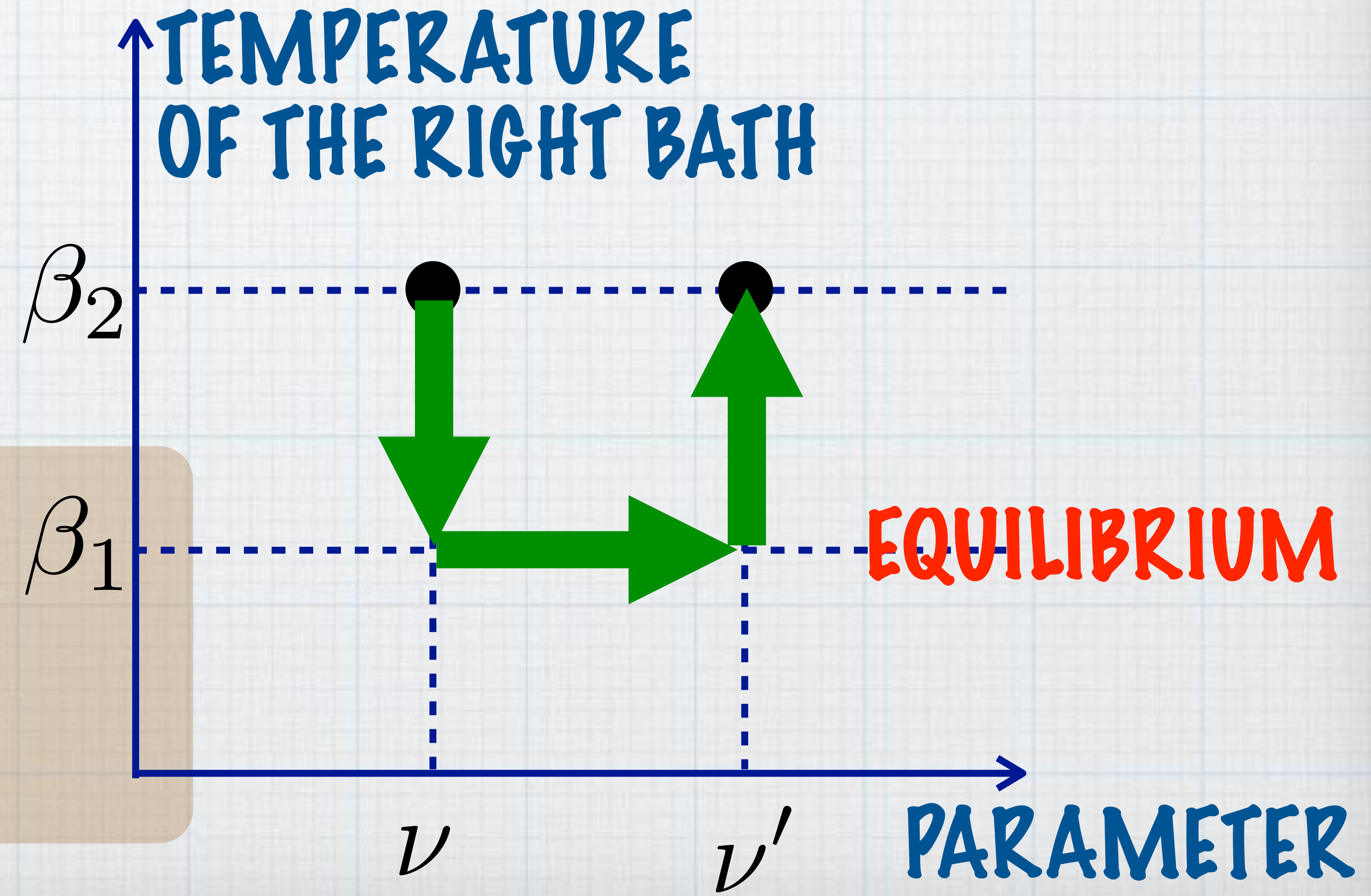
USE THE COMBINATION OF THE THREE PROCESSES

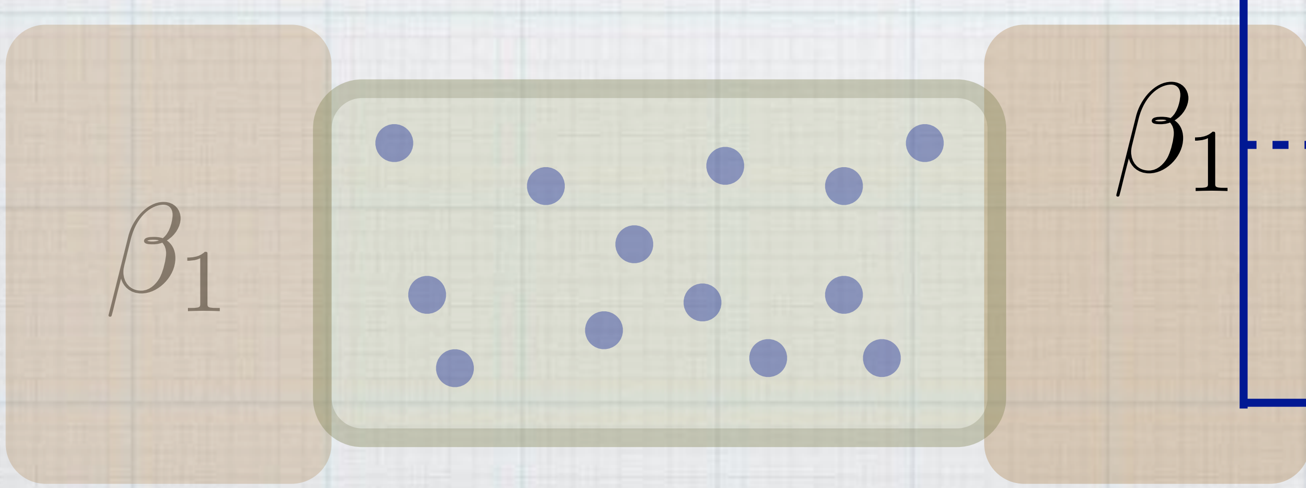
$$(\beta_1, \beta_2, \nu) \xrightarrow{\mathrm{a}} (\beta_1, \beta_1, \nu) \xrightarrow{\mathrm{b}} (\beta_1, \beta_1, \nu') \xrightarrow{\mathrm{c}} (\beta_1, \beta_2, \nu')$$

FROM THE EXTENDED CLAUSIUS RELATION, ONE GETS

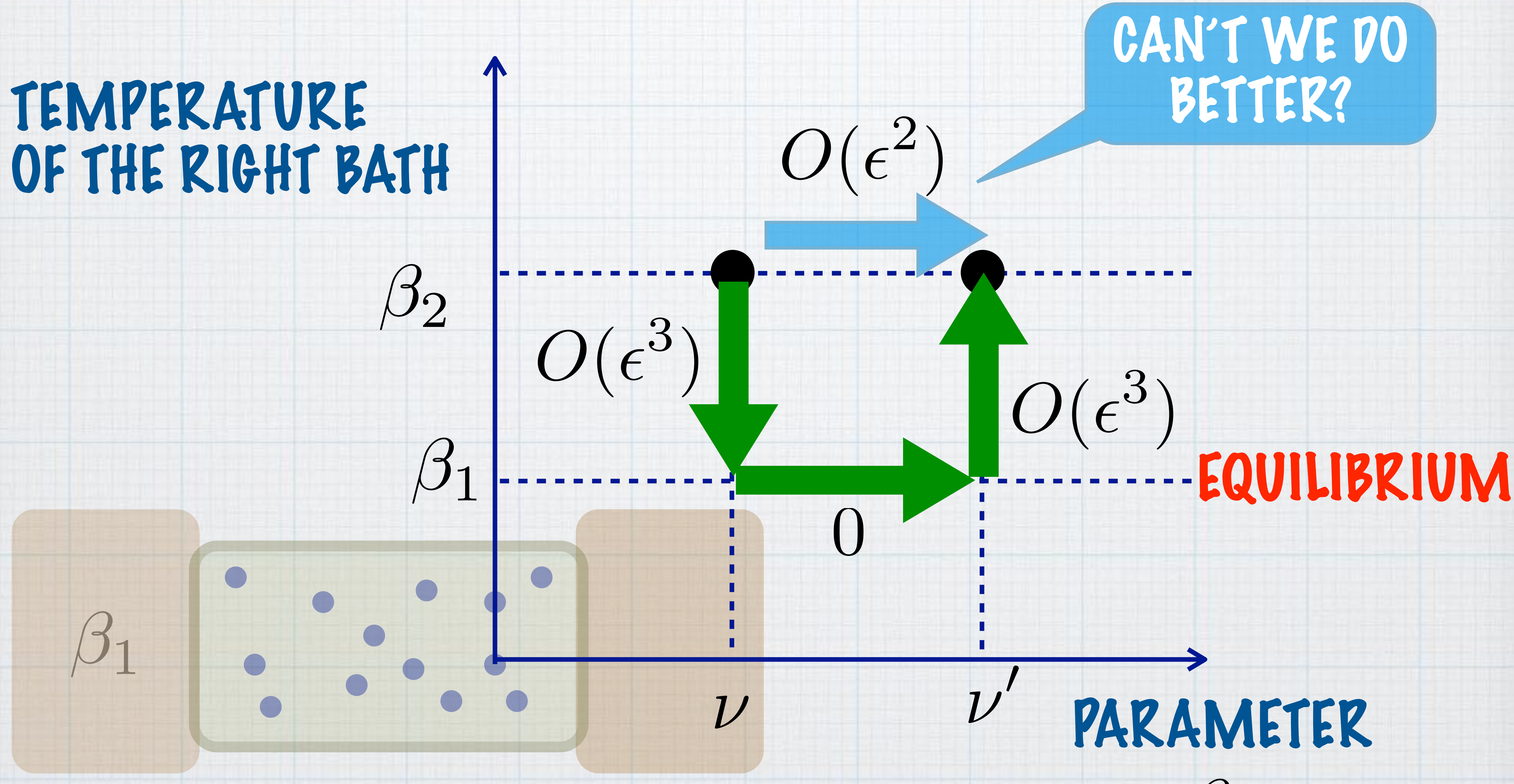
$$S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu)$$
$$= -\langle \Theta_{\mathrm{ex}} \rangle_{\mathrm{a}} - \beta_1 \, \Delta Q_{\mathrm{b}} - \langle \Theta_{\mathrm{ex}} \rangle_{\mathrm{c}} + O(\epsilon^3)$$

WE CAN DETERMINE THE DIFFERENCE WITH THE PRECISION OF $O(\epsilon^2)$





POSSIBLE ERROR IN EACH PROCESS

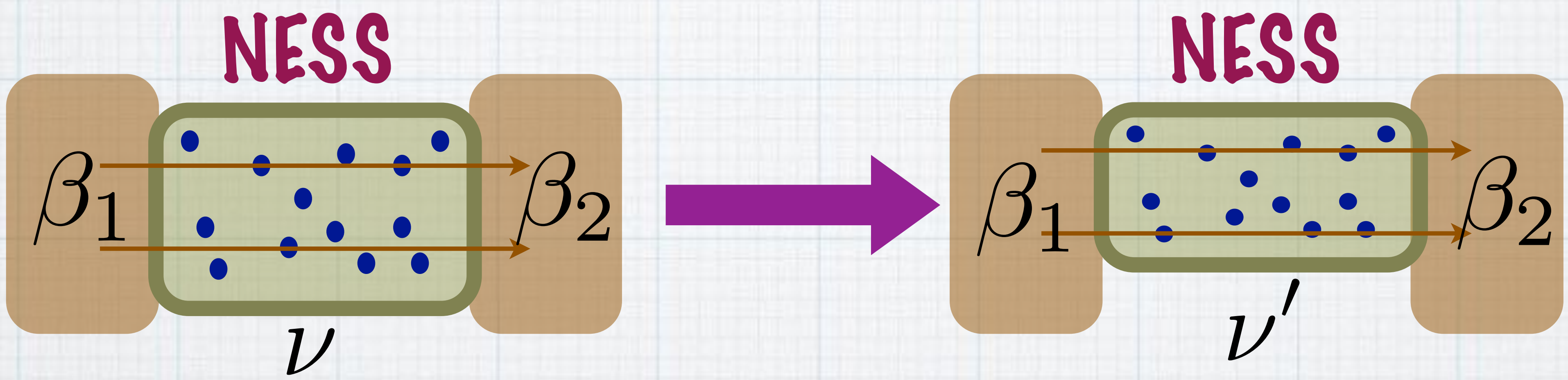


THE TEMPERATURE OF THE LEFT BATH IS FIXED AT β_1

NONLINEAR NONEQULIBRIUM RELATION

THE SECOND ORDER EXTENDED CLAUSIUS RELATION

FOR THE DIRECT PATH FROM (β_1, β_2, ν) TO (β_1, β_2, ν')



$$S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu) = -\langle \Theta_{\rm ex}^{\hat\alpha} \rangle^{\hat\alpha} + \frac{\beta_1 + \beta_2}{4} \langle W^{\hat\alpha}; \Theta^{\hat\alpha} \rangle^{\hat\alpha} + O(\epsilon^3 \delta)$$

$$\delta = |\nu' - \nu|$$

W WORK DONE TO THE SYSTEM

$$\langle W ; \Theta \rangle := \langle W\, \Theta \rangle - \langle W \rangle \langle \Theta \rangle$$

THE SECOND ORDER EXTENDED CLAUSIUS RELATION

$$S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu)$$
$$= -\langle \Theta_{\mathrm{ex}}^{\hat{\alpha}} \rangle^{\hat{\alpha}} + \frac{\beta_1 + \beta_2}{4} \langle W^{\hat{\alpha}}; \Theta^{\hat{\alpha}} \rangle^{\hat{\alpha}} + O(\epsilon^3 \delta)$$

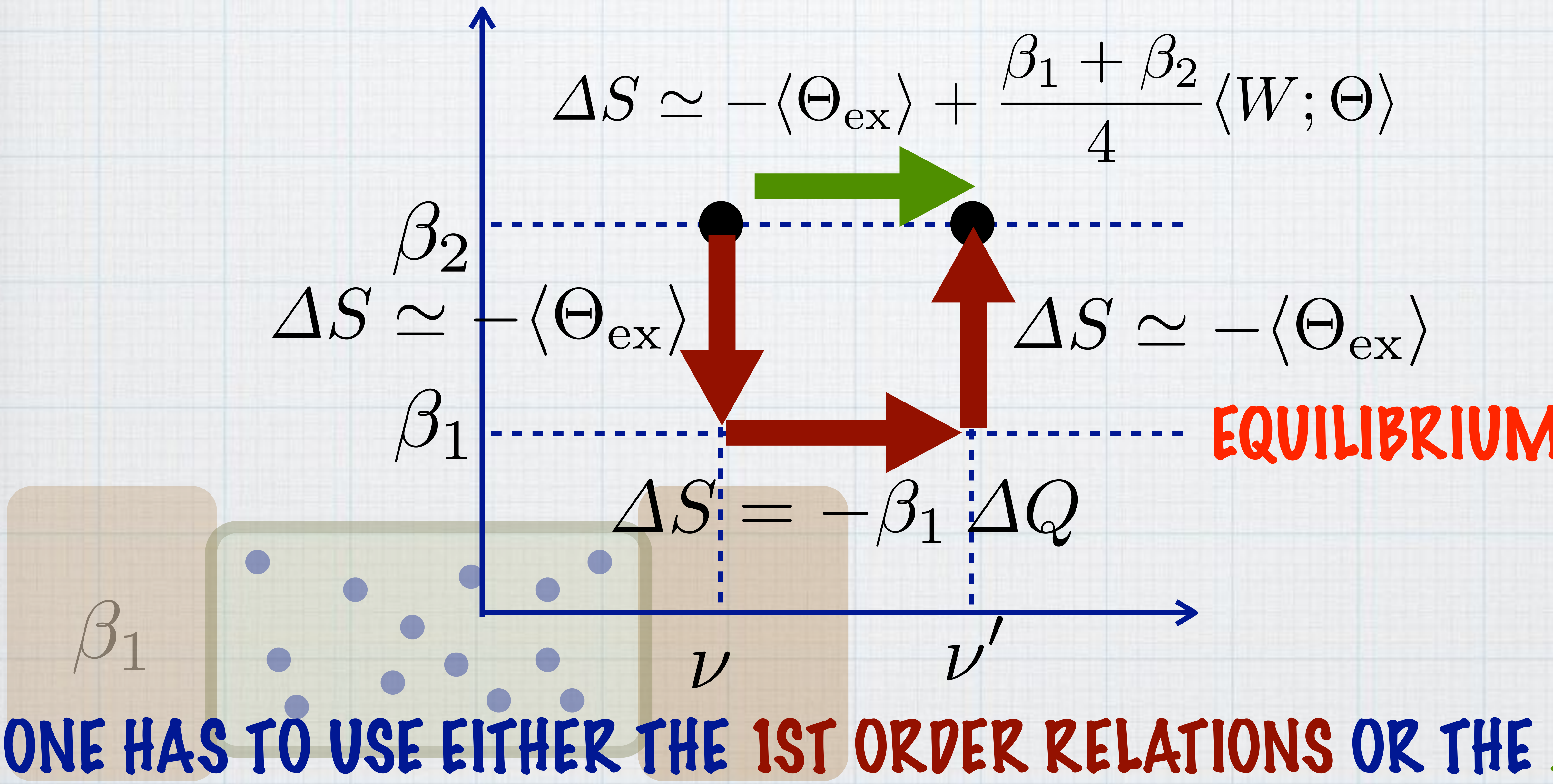
THE RELATION TAKES INTO ACCOUNT "NONLINEAR NONEQUILIBRIUM" CONTRIBUTIONS, AND HAS A DESIRED HIGHER PRECISION.

BUT IT CONTAINS A CORRELATION BETWEEN HEAT AND WORK.

IT IS A RELATION BETWEEN MACROSCOPIC QUANTITIES; BUT CAN WE CALL IT A THERMODYNAMIC RELATION?

OPERATIONAL DETERMINATION OF ENTROPY

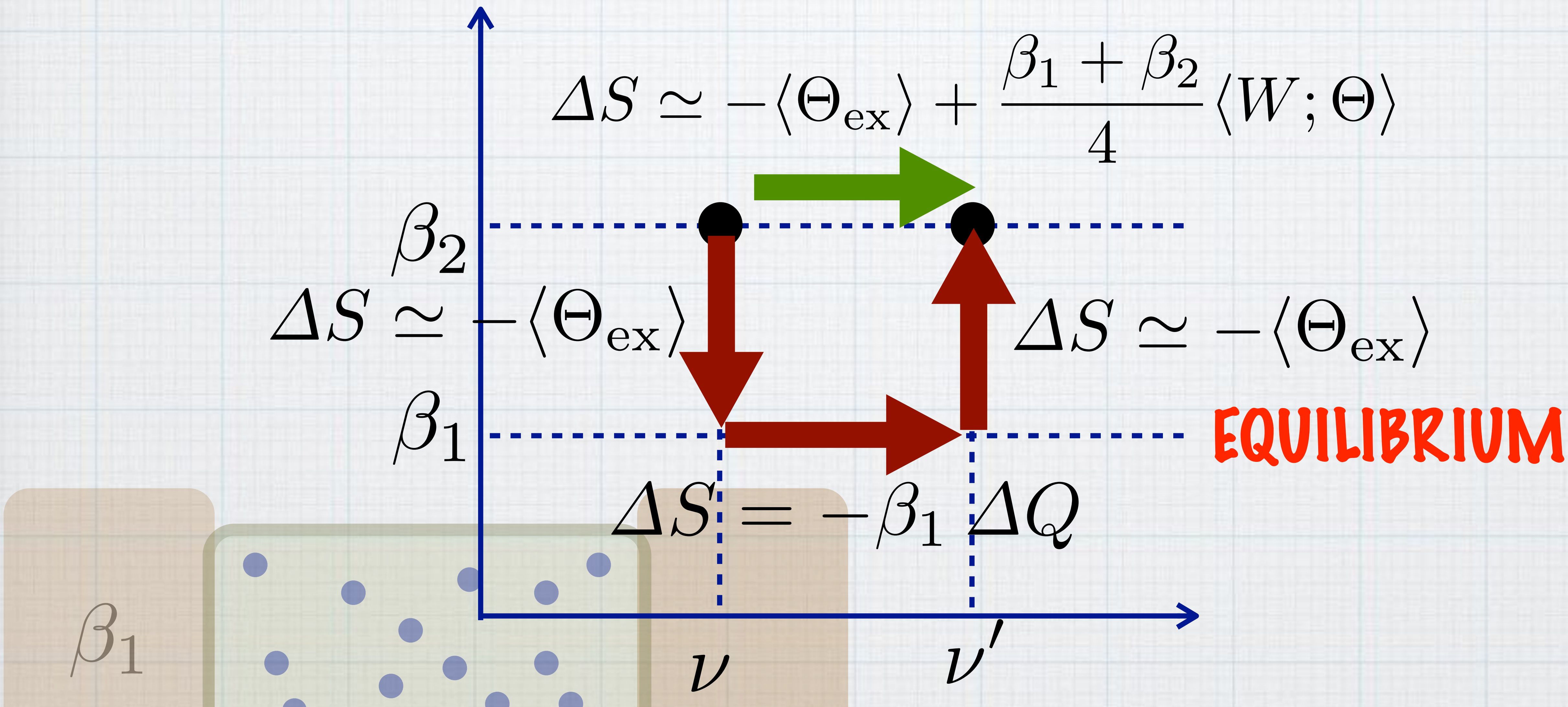
DETERMINE $S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu)$ TO THE ORDER $O(\epsilon^2)$



ONE HAS TO USE EITHER THE 1ST ORDER RELATIONS OR THE 2ND ORDER RELATION, DEPENDING ON THE PATHS!

THE SECOND “TWIST” IN SST

DETERMINE $S(\beta_1, \beta_2, \nu') - S(\beta_1, \beta_2, \nu)$ TO THE ORDER $O(\epsilon^2)$



ONE HAS TO USE EITHER THE 1ST ORDER RELATIONS OR THE 2ND ORDER RELATION, DEPENDING ON THE PATHS!

SUMMARY

- ☑ OUR RESULTS ARE MATHEMATICALLY RIGOROUS FOR MARKOV JUMP PROCESSES, BUT NOT ENTIRELY RIGOROUS FOR OTHER MODELS
- ☑ WE FOUND A NATURAL EXTENSION OF CLAUSIUS RELATION FOR OPERATIONS BETWEEEN NESS, WHICH ENABLES ONE TO OPERATIONALLY DETERMINE NONEQULIBRIUM ENTROPY TO THE SECOND ORDER IN $\epsilon = |\beta_1 - \beta_2|$
- ☑ THE NONEQUILIBRIUM ENTROPY HAS AN EXPRESSION IN TERMS OF SYMMETRIZED SHANNON ENTROPY

$$S_{\rm sym}[\rho] := -\int dx\, \rho_x \, \log \sqrt{\rho_x\, \rho_{x^*}}$$

WHAT DOES IT MEAN??

- IS THERE A MEANINGFUL THERMODYNAMICS FOR NESS WHICH YIELDS NONTRIVIAL EXPERIMENTAL PREDICTIONS?
- THERE ARE MANY DIFFERENT ATTEMPTS, E.G., BY Netochny-Maes, Jona-Lasinio et al., Nakagawa-Sasa, ...