Lieb-Schultz-Mattis theorem and its variations for quantum spin chains

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subject: mathematical physics of quantum spin systems

prerequisite: basics of quantum spins

contents: original Lieb-Schultz-Mattis (LSM) theorem
time-reversal of a spin and the index for edges states
proof of LSM type theorems based on the index

2π-rotations of a spin

single spin $\hat{\boldsymbol{S}} = (\hat{S}^x, \hat{S}^y, \hat{S}^z)$, $[\hat{S}^x, \hat{S}^y] = i\hat{S}^z$ etc.

$$\boxed{\hat{\boldsymbol{S}}^2} = S(S+1), \quad S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

$S \rightarrow$ spin quantum number

$\exp[-i\theta \hat{S}^\alpha]$ θ-rotation about the α-axis

$\alpha = x, y, z$

$\theta = 2\pi$

$$\exp[-i2\pi\hat{S}^\alpha] = \begin{cases} \hat{I} & S = 1, 2, \ldots \\ -\hat{I} & S = \frac{1}{2}, \frac{3}{2}, \ldots \end{cases}$$

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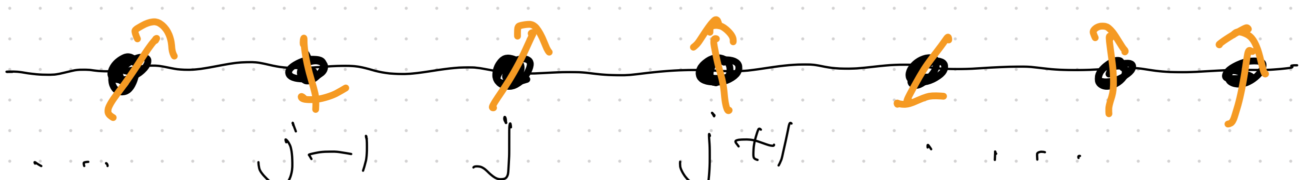
Lieb-Schultz-Mattis Type theorem

a no-go theorem which states that certain quantum many-body systems cannot have a unique ground state with a nonzero energy gap

symmetry $\longrightarrow$ low energy properties

quantum spin chains



Part I

the original Lieb-Schultz-Mattis theorem

The original LSM theorem

Lieb, Schultz, Mattis 1961
Affleck, Lieb 1986

▷ antiferromagnetic Heisenberg chain

$$\hat{H} = \sum_{j=1}^{L} \hat{\mathbf{S}}_j \cdot \hat{\mathbf{S}}_{j+1}$$

$$\hat{\mathbf{S}}_j = (\hat{S}_j^x, \hat{S}_j^y, \hat{S}_j^z)$$

$$\hat{\mathbf{S}}_j^2 = S(S+1), \quad S = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots$$

E_{GS}: g.s. energy

THEOREM if $S = \frac{1}{2}, \frac{3}{2}, \cdots$ there exists an energy eigenvalue E such that

$$E_{GS} < E \leq E_{GS} + \frac{C}{\ell}$$

for any $\ell < L$

$$C = 8\pi^2 S^2$$

outline of the proof

the g.s. $|GS\rangle$ of $\hat{H}$ is unique for $\forall$ even L

(Marshall 1955, Lieb, Mattis 1962)

(1) variational estimate

U(1) invariance

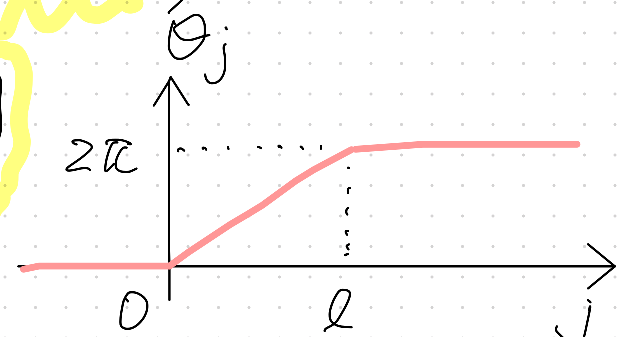
uniqueness + rotation invariance of $\hat{H}$

$$\rightarrow \exp\left[-i\sum_{j=1}^{L} \theta \hat{S}_j^z\right]|GS\rangle = |GS\rangle$$

gradual non-uniform rotation (twist) by θ_j

$$\boxed{\hat{U}_\ell = \exp\left[-i\sum_{j=1}^{\ell} \theta_j \hat{S}_j^z\right]}$$

$$\theta_j = \frac{2\pi}{\ell} j = \Delta\theta\, j$$



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(1) variational estimate

twist operator $\hat{U}_\ell = \exp\left[-i \sum_{j=1}^{\ell} \Delta\theta\, j\, \hat{S}_j^z\right]$

$$\boxed{\Delta\theta = \frac{2\pi}{\ell}}$$

variational state $|\Psi_\ell\rangle = \hat{U}_\ell |GS\rangle$

Bloch

$0 \leq j < \ell$ $\quad \underbrace{\langle \Psi_\ell | \hat{\mathbf{S}}_j \cdot \hat{\mathbf{S}}_{j+1} | \Psi_\ell \rangle}$ $= \frac{E_{GS}}{L} + O((\Delta\theta)^2)$

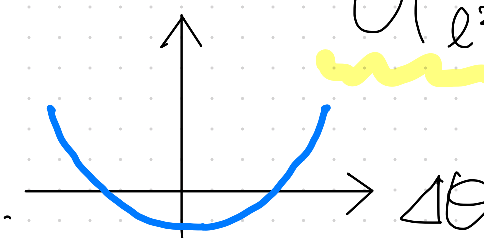
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad O\left(\frac{1}{\ell^2}\right)$

thus

$$\langle \Psi_\ell | \hat{H} | \Psi_\ell \rangle - E_{GS} \leq \boxed{\frac{C}{\ell}}$$

valid for any $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$



(2) orthogonality $\quad l$: even

unitary $\hat{R}$ s.t. $\hat{R}^\dagger \hat{S}_j^\alpha \hat{R} = \begin{cases} \hat{S}_{l-j}^\alpha & \alpha = x \\ -\hat{S}_{l-j}^\alpha & \alpha = y, z \end{cases}$

▷ $\hat{R}^\dagger \hat{H} \hat{R} = \hat{H}$, $|GS\rangle$ is the unique g.s. $\rightarrow$ $\hat{R}|GS\rangle = \pm|GS\rangle$

▷ $\hat{U}_l = \exp\left[-i \sum_{j=0}^{l} \frac{2\pi}{l} j \hat{S}_j^z\right]$

$$\hat{R}^\dagger \hat{U}_l \hat{R} = \exp\left[i \sum_{j=0}^{l} \frac{2\pi}{l} (l-j) \hat{S}_j^z\right] = \underbrace{\exp\left[i 2\pi \sum_{j=0}^{l} \hat{S}_j^z\right]}_{=-1} \hat{U}_l$$

if $S = \frac{1}{2}, \frac{3}{2}, \cdots$

$$\langle GS|\Psi_l\rangle = \langle GS|\hat{U}_l|GS\rangle = \langle GS|\hat{R}^\dagger \hat{U}_l \hat{R}|GS\rangle$$

$$= -\langle GS|\hat{U}_l|GS\rangle = 0$$

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completing the proof

$|GS\rangle$ the unique g.s. of $\hat{H} = \sum_{j=1}^{L} \hat{\mathbf{S}}_j \cdot \hat{\mathbf{S}}_{j+1}$

$|\Psi_\ell\rangle = \hat{U}_\ell |GS\rangle$ $\quad \hat{U}_\ell$: gradual "twist"

(1) variational estimate

$$\langle \Psi_\ell | \hat{H} | \Psi_\ell \rangle - E_{GS} \leq \frac{C}{\ell} \quad \text{for any } \ell < L$$

(2) orthogonality

if ℓ is even and $S = \frac{1}{2}, \frac{3}{2}, \cdots$; $\langle GS | \Psi_\ell \rangle = 0$

then there is an energy eigenstate $|\tilde{\Psi}_\ell\rangle$ with energy eigenvalue E_ℓ s.t. $E_\ell - E_{GS} \leq \frac{C'}{\ell}$ //

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remarks

THEOREM if $S = \frac{1}{2}, \frac{3}{2}, \cdots$ $\exists$ an energy eigenvalue E s.t. $0 < E - E_{GS} \lesssim \frac{C}{l}$ for any $l < L$

(i) in the $L \uparrow \infty$ limit there are two possibilities:

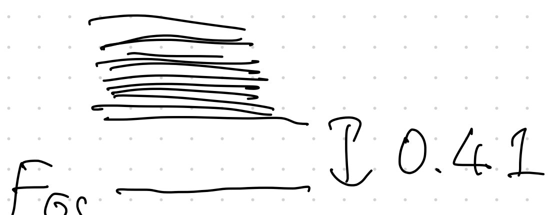
(i) unique g.s. with gapless excitations ← this is the case.

(ii) there are mulitiple g.s.

(ii) no information if $S = 1, 2, \cdots$

in fact the model has a unique g.s. with a gap!

$S = 1$



Haldane gap

Part II

new LSM-type theorem

LSM theorem and early extensions

Lieb, Schultz, Mattis 1961, Affleck, Lieb 1986

Oshikawa, Yamanaka, Affleck 1997

Oshikawa 2000, Hasting 2000, Nachtergaele, Sims 2007.

U(1) invariance is essential!

recent extensions

Chen, Gu, Wen 2011, Watanabe, Po, Vishwanath, Zaletel 2013

similar no-go theorems for model with only discrete symmetry

closely related to "topological" condensed matter physics

THEOREM take a quantum spin chain with $S=\frac{1}{2},\frac{3}{2},\cdots$ and a short-ranged translation invariant Hamiltonian that is invariant under time-reversal $\hat{S}_j^\alpha \to -\hat{S}_j^\alpha$ $\alpha=x,y,z$

it can never be the case that the infinite volume g.s. is unique and accompanied by a nonzero gap.

example $$\hat{H}=\sum_j\left\{J_x\hat{S}_j^x\hat{S}_{j+1}^x+J_y\hat{S}_j^y\hat{S}_{j+1}^y+J_z\hat{S}_j^z\hat{S}_{j+1}^z\right\}$$

Chen, Gu, Wen 2011

Watanabe, Po, Vishwanath, Zaletel 2013 proof for MPS

Ogata, Tasaki 2019 full theorem

Ogata, Tachikawa, Tasaki 2020 new general proof

Matsui 2001 essential argument

strategy of our proof

Y. Ogata, Y. Tachikawa, H. Tasaki
General Lieb-Schultz-Mattis type theorems for quantum spin chains
arXiv:2004:06458

Oshikawa 2000

- examine a necessary condition for the existence of a unique gapped g.s.
- make full use of the index for "edge state" developed in the study of symmetry protected topological phase

time-reveral for a single spin $\hat{\mathbf{S}} = (\hat{S}^x, \hat{S}^y, \hat{S}^z)$

▷ time-reveral map Γ (antilnear $*$-automorphism)

$$\Gamma(\hat{S}^\alpha) = -\hat{S}^\alpha \quad (\alpha = x, y, z)$$

$$\Gamma(\hat{A}\hat{B}) = \Gamma(\hat{A})\Gamma(\hat{B}), \quad \Gamma(\hat{A}^\dagger) = \Gamma(\hat{A})^\dagger$$

$$\Gamma(\alpha\hat{A} + \beta\hat{B}) = \alpha^* \Gamma(\hat{A}) + \beta^* \Gamma(\hat{B})$$

why antilinear?

$$\hat{S}^x\hat{S}^y - \hat{S}^y\hat{S}^x = i S^z$$

$$\Gamma \downarrow$$

$$\hat{S}^x\hat{S}^y - \hat{S}^y\hat{S}^x = -i\,(-S^z)$$

time-reveral for a singlespin

- time-reveral antiunitary operator $\hat{\Theta} \to \Gamma(\hat{A}) = \hat{\Theta}^{-1}\hat{A}\hat{\Theta}$

$$\hat{\Theta} = e^{-i\pi\hat{S}^y}\hat{K} = \hat{K}e^{-i\pi\hat{S}^y}$$

π-rotation about y-axis
all the matrix entries are real

complex conjugation

$$\hat{K}\begin{pmatrix}\varphi_s \\ \vdots \\ \varphi_{-s}\end{pmatrix} = \begin{pmatrix}\varphi_s^* \\ \vdots \\ \varphi_{-s}^*\end{pmatrix}$$

standard basis

- $$\hat{\Theta}^2 = \hat{K}^2\left(e^{-i\pi\hat{S}^y}\right)^2 = e^{-i2\pi\hat{S}^y} = \begin{cases}\hat{\mathbb{1}} & S=1,2,\cdots \\ -\hat{\mathbb{1}} & S=\frac{1}{2},\frac{3}{2},\cdots\end{cases}$$

index $I \in \{0,1\} = H^2(\mathbb{Z}_2, U(1)_p)$

Projective representation

$$\hat{\Theta}^2 = (-1)^I\hat{\mathbb{1}} \qquad I = \begin{cases}0 & S=1,2,\cdots \\ 1 & S=\frac{1}{2},\frac{3}{2},\cdots\end{cases}$$

$I_1 \quad I_2$

$I_1 + I_2 \pmod 2$

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remark: Kramers degeneracy

if $S = \frac{1}{2}, \frac{3}{2}, \ldots$

any state $|\varphi\rangle$ and $\hat{\Theta}|\varphi\rangle$ are orthogonal

if $\hat{\Theta}^{-1}\hat{H}\hat{\Theta} = \hat{H}$ ← time-reversally invariant

$\hat{H}|\varphi\rangle = E|\varphi\rangle$ implies

$$\hat{H}\hat{\Theta}|\varphi\rangle = \hat{\Theta}\hat{\Theta}^{-1}\hat{H}\hat{\Theta}|\varphi\rangle = \hat{\Theta}\hat{H}|\varphi\rangle = E\hat{\Theta}|\varphi\rangle$$

any eigenvalue must be degenerate

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no unique g.s. ← LSM-like statement for a single spin

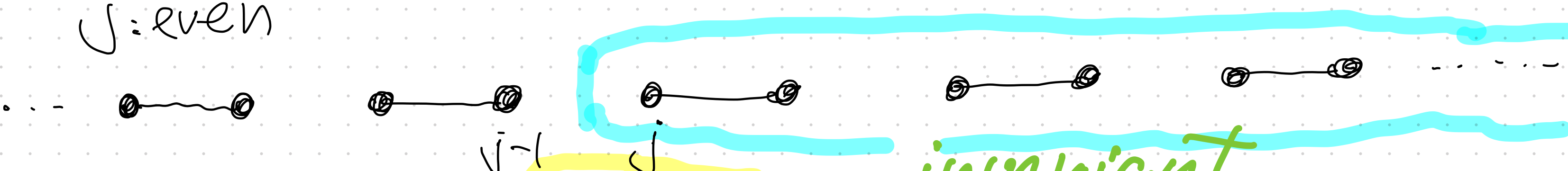
indices for edge states: an example

(iii) infinite $S=\frac{1}{2}$ chain

dimer state $\displaystyle\bigotimes_{j:\text{even}} \frac{1}{\sqrt{2}}\left\{|\uparrow\rangle_j|\downarrow\rangle_{j+1} - |\downarrow\rangle_j|\uparrow\rangle_{j+1}\right\}$

↳ time-reversally invariant

j: even



$\boxed{\mathcal{I}_j = 0}$

invariant

j: odd

$\boxed{\mathcal{I}_j = 1}$

transforms as a $S=1/2$!?

indices for edge states: examples

▷ infinte $S=1$ chain   zero-state $\boxed{\bigotimes_j |0\rangle_j}$

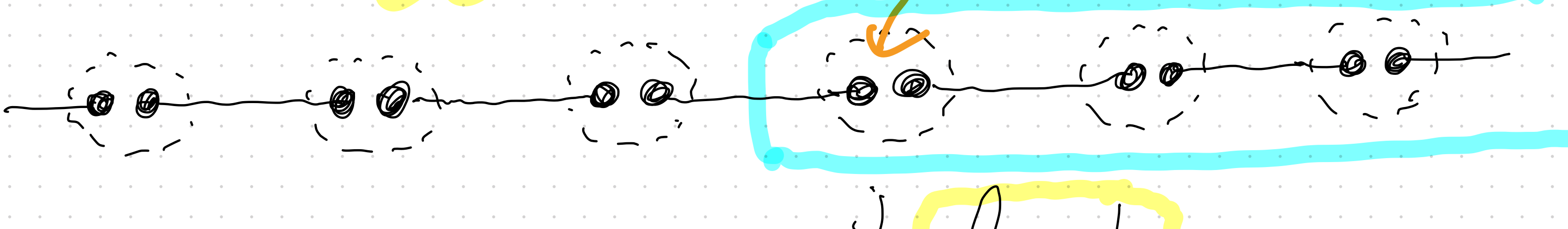
$$\cdots 0 - 0 - 0 - 0 - \underbrace{0}_{j} - 0 - 0 - \cdots$$

$$\boxed{\mathcal{J}_j = 0}$$

▷ infinte $S=1$ chain

AKLT state

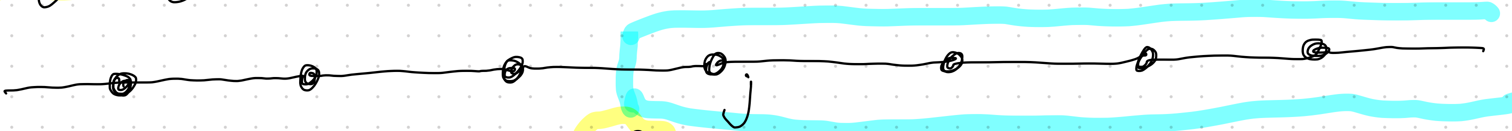
effective $S=1/2$ edge spin



$$\boxed{\mathcal{J}_j = 1}$$

indices for edge states: general results

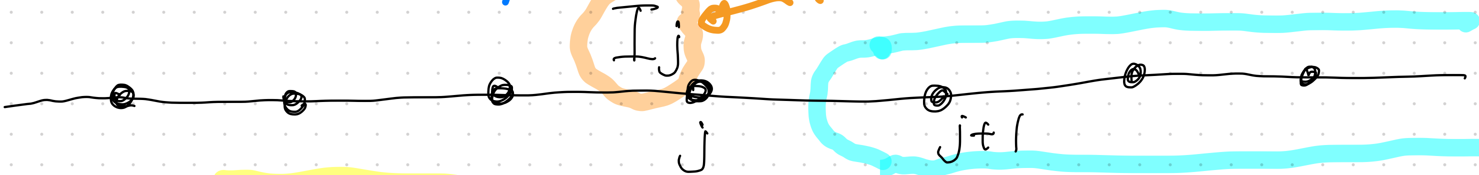
- general short-ranged time-reversally invariant Hamiltonian
- unique ground state accompanied by a nonzero gap



$\mathcal{I}_j$ well-defined index $\in \{0,1\}$

• essential identity

I_j index for the spin at j



$$\mathcal{I}_j = I_j + \mathcal{I}_{j+1}$$

$\mathcal{I}_{j+1}$

remarks about the index

- indices for MPS (matrix product states)

  Pérez-García, Wolf, Sanz, Verstraete, Cirac 2008

  Pollmann, Turner, Berg, Oshikawa 2010

- indices for general unique gapped g.s. with symmetry

  based on a series of highly nontrivial mathematical works

  Matsui 2001, 2013, Hastings 2007

  Ogata 2018 and more

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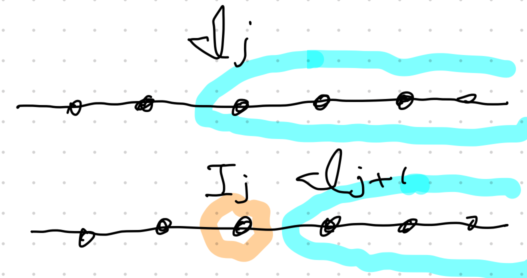
C^*-algebra
split state
GNS construction
type-I factor
projective repre

proof of the LSM-type theorem

- assume that there is a time-reversally invariant unique gapped g.s.

$$\triangleleft_j = I_j + \triangleleft_{j+1}$$



- assume translation invariance

$$\triangleleft_j = \triangleleft_{j+1} \Rightarrow \boxed{I_j = 0} \Rightarrow \boxed{S = 1, 2, 3, \cdots}$$

- $\boxed{S = \frac{1}{2}, \frac{3}{2}, \cdots} \Rightarrow$ time rev. & transl. invariant unique gapped g.s. is impossible!!

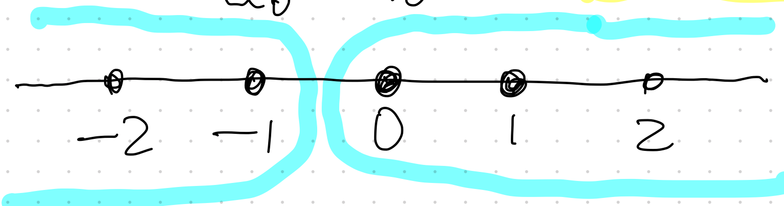
THEOREM take a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \ldots$ and a short-ranged translation invariant Hamiltonian that is invariant under time-reversal $\hat{S}_j^\alpha \to -\hat{S}_j^\alpha$ $\alpha = x, y, z$

it can never be the case that the infinite volume g.s. is unique and accompanied by a nonzero gap.

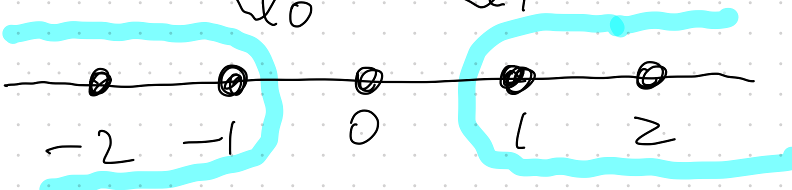
reflection invariant chain

invariant under $\begin{cases} \text{time-reversal} \\ \text{reflection } j \to -j \end{cases}$

$-\ell_0 \quad \ell_0$



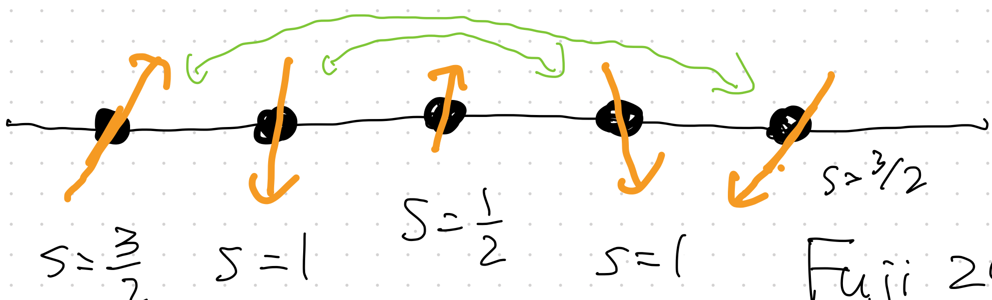
$-\ell_0 = \ell_1$



$\ell_0 = I_0 + \ell_1 \Rightarrow I_0 = \ell_0 - \ell_1 = 2\ell_0 \Rightarrow I_0 = 0$

spin at the origin is $S_0 = 1, 2, \ldots$

THEOREM take a quantum spin chain with site-dependent spin and a short-range Hamiltonian that is invariant under reflection $j \to -j$ and time-reversal if the spin at the origin satisfies $S_0 = \frac{1}{2}, \frac{3}{2}, \cdots$ then it can never be the case that the infinite volume g.s. is unique and accompanied by a nonzero gap.



Fuji 2014

Po, Watanabe, Jian, Zaletel 2017

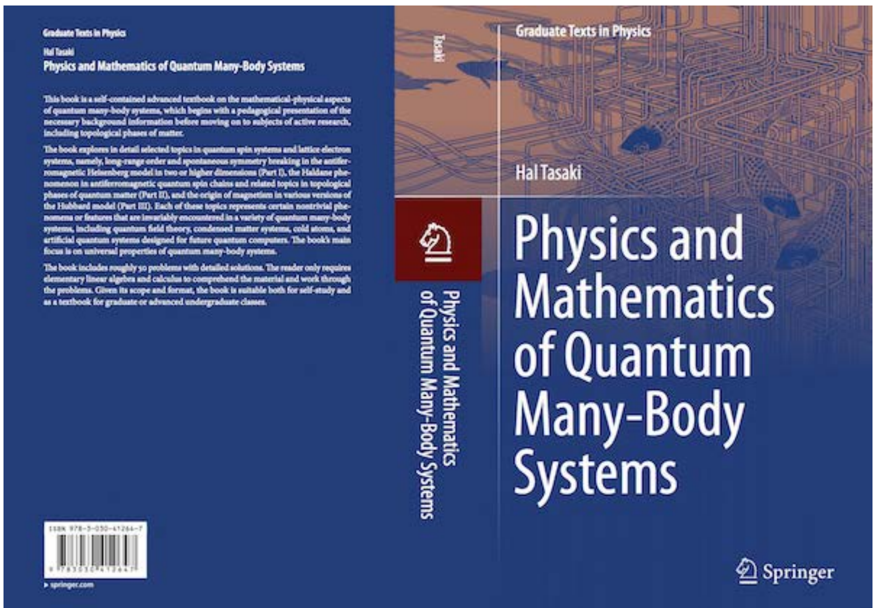
Summary + discussion

- the original LSM theorem makes use of the "gradual twist" based on the $U(1)$ symmetry
- new LSM-type theorems follow from the properties of indices for "edge states"
- the results extend to general symmetry group G. then $\{0,1\}$ is replaced by the degree-2 group cohomology $H^2(G, U(1)_\rho)$

advertisement

for background, related topic,

and more, see



Hal Tasaki

"Physics and Mathematics of Quantum Many-Body Systems"

Springer June 2020

with four imginative illustrations by Mari Okazaki