Symmetry-protected topological (SPT) phases and topological indices in quantum spin chains

part 2: Ogata index and its applications to SPT and full SPT order

Hal Tasaki

online lecture @ YouTube / August 2021

Index for
projective representation
of a symmetry group

Representation and projective representation of a group

G a finite group (for the symmetry of the model)

representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = U_{gh}$

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

$$\varphi(g,h) \in \mathrm{U}(1) := \{z \in \mathbb{C} \,|\, |z| = 1\}$$

associativity $U_f(U_g U_h) = (U_f U_g) U_h$ implies

$$\frac{\varphi(g,h)\,\varphi(f,gh)}{\varphi(f,g)\,\varphi(fg,h)} = 1 \quad \text{for any } f,g,h \in G$$

φ is a 2-cocycle

$\mathrm{Z}^2(G, \mathrm{U}(1))$ the set of all 2-cocycles $\varphi : G \times G \to \mathrm{U}(1)$

Remark: we can treat proj. reps. with antiuniatary operators

Index for a projective representation

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

equivalent projective representation

$$U'_g = \psi(g)\, U_g \qquad \psi(g) \in \mathrm{U}(1) := \{z \in \mathbb{C} \,|\, |z| = 1\}$$

$$U'_g U'_h = \varphi'(g,h)\, U'_{gh} \qquad \varphi'(g,h) = \frac{\psi(g)\psi(h)}{\psi(gh)}\varphi(g,h)$$

$\varphi \sim \varphi'$ **iff** $\varphi'(g,h) = \dfrac{\psi(g)\psi(h)}{\psi(gh)}\varphi(g,h)$ **with some** $\psi(g)$

second group cohomology $\mathrm{H}^2(G, \mathrm{U}(1)) = \mathrm{Z}^2(G, \mathrm{U}(1))/\sim$

← abelian group

$\mathrm{H}^2(G, \mathrm{U}(1)) \ni \mathrm{ind}$ **characterizes an equivalence class of the projective representations of** G

Index for a projective representation

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

$\mathrm{H}^2(G, \mathrm{U}(1)) \ni \mathrm{ind}$ characterizes an equivalence class of the projective representations of G

the indices can be added!

two projective representations

$$u_g^{(1)} \qquad u_g^{(1)} u_h^{(1)} = \varphi_1(g,h)\, u_{gh}^{(1)} \qquad \mathrm{ind}_1$$

$$u_g^{(2)} \qquad u_g^{(2)} u_h^{(2)} = \varphi_2(g,h)\, u_{gh}^{(2)} \qquad \mathrm{ind}_2$$

$U_g = u_g^{(1)} \otimes u_g^{(2)}$ proj. rep with $\varphi(g,h) = \varphi_1(g,h)\, \varphi_2(g,h)$

$$\mathrm{ind} = \mathrm{ind}_1 + \mathrm{ind}_2$$

Important example $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{e}, \mathrm{x}, \mathrm{y}, \mathrm{z}\}$$

$$\mathrm{x}^2 = \mathrm{y}^2 = \mathrm{z}^2 = \mathrm{e} \quad \mathrm{xy} = \mathrm{yx} = \mathrm{z} \quad \mathrm{yz} = \mathrm{zy} = \mathrm{x} \quad \mathrm{zx} = \mathrm{xz} = \mathrm{y}$$

$$\mathrm{H}^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2 = \{0, 1\} \ni \mathrm{ind}$$

proj. rep. on a single spin $\boldsymbol{S} = (S^{\mathrm{x}}, S^{\mathrm{y}}, S^{\mathrm{z}}) \qquad \boldsymbol{S}^2 = S(S+1)$

$$U_{\mathrm{e}} = I \qquad U_g = \exp[-i\pi S^g] \qquad g \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$$

$$U_{\mathrm{x}} U_{\mathrm{y}} = U_{\mathrm{z}} \quad U_{\mathrm{y}} U_{\mathrm{z}} = U_{\mathrm{x}} \quad U_{\mathrm{z}} U_{\mathrm{x}} = U_{\mathrm{y}}$$

integer spin $(S = 1, 2, \ldots)$ **genuine representation**

$$(U_g)^2 = I \qquad U_g U_h = U_h U_g$$

half-odd-integer spin $(S = \frac{1}{2}, \frac{3}{2}, \ldots)$

$$(U_g)^2 = -I \quad U_g U_h = -U_h U_g$$

$g, h \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$

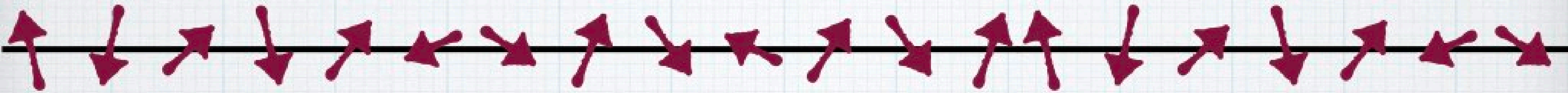
$g \neq h$

nontrivial proj. rep.

$$\mathrm{ind} = \begin{cases} 0 & S = 1, 2, \ldots \quad \textbf{trivial} \\ 1 & S = \frac{1}{2}, \frac{3}{2}, \ldots \quad \textbf{nontrivial} \end{cases}$$

Toward index for a unique gapped ground state

General quantum spin chain



$\mathfrak{h}_j$ **Hilbert space at site** $j \in \mathbb{Z}$ $\qquad \dim(\mathfrak{h}_j) \le d_0$

local operator = operator that acts nontrivially on $\bigotimes_{j\in S} \mathfrak{h}_j$ with finite $S \subset \mathbb{Z}$

C^***-algebra** $\mathfrak{A} = \overline{\{\text{all local operators}\}}$

G **symmetry group (finite group)**

$u_g^{(j)}$ **unitary on** $\mathfrak{h}_j$

projective representation with index $\mathrm{ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

***-automorphism on** $\mathfrak{A}$

$$\Xi_g(A) = \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right) A \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right)^*$$

for $g \in G$ **and a local operator** A

$$\Xi_g \circ \Xi_h = \Xi_{gh}$$

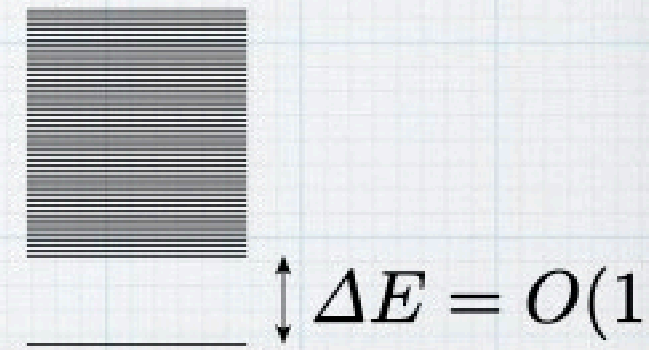
G-invariant Hamiltonian and a unique gapped g.s.

formal expression

G-invariant short ranged Hamiltonian $H = \sum_{j\in\mathbb{Z}} h_j$

$h_j = h_j^*$ acts only on $\bigotimes_{k;|k-j|\le r_0} \mathfrak{h}_k$

$\Xi_g(h_j) = h_j$ for any $j \in \mathbb{Z}$ and $g \in G$



basic assumption: the ground state ω of H is unique and accompanied by a nonzero energy gap

$$\omega(A) = \lim_{L\uparrow\infty} \langle \Phi_{\mathrm{GS}}^{(L)}, A\Phi_{\mathrm{GS}}^{(L)} \rangle$$

Def: a state is a linear function $\omega : \mathfrak{A} \to \mathbb{C}$ such that $\omega(I) = 1$ and $\omega(A^*A) \ge 0$ for any $A \in \mathfrak{A}$

Def: ω is a g.s. if $\omega(A^*[H, A]) \ge 0$ for any local operator A

Def: a unique g.s. ω is accompanied by a nonzero gap if there is $\gamma > 0$ and $\omega(A^*[H, A]) \ge \gamma\, \omega(A^*A)$ for any A s.t. $\omega(A) = 0$

G-invariant Hamiltonian and a unique gapped g.s.

formal expression

G-invariant short ranged Hamiltonian $H = \sum_{j\in\mathbb{Z}} h_j$

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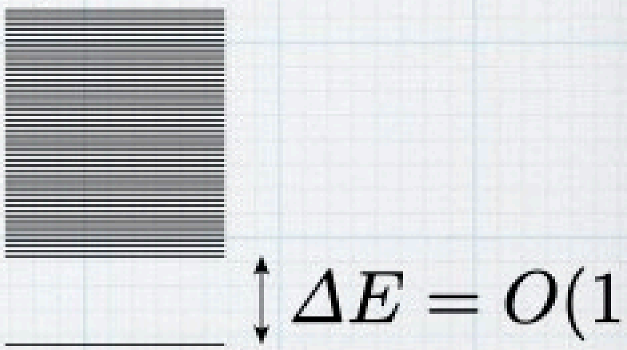


basic assumption: the ground state ... of H is unique and accompani...

if $H|\Phi\rangle = E|\Phi\rangle$

$\langle\Phi|A^*[H,A]|\Phi\rangle \ge 0$ implies

$$\frac{\langle\Phi|A^*\,H\,A|\Phi\rangle}{\langle\Phi|A^*A|\Phi\rangle} \ge E$$

... $\langle\Phi_{\mathrm{GS}}^{(L)}, A\Phi_{\mathrm{GS}}^{(L)}\rangle$

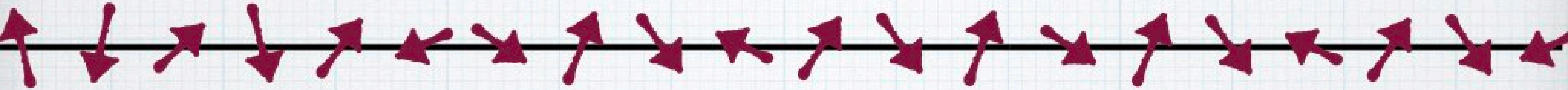
Def: a state ... such that

$\omega(I) = 1$ and ... for any $A \in \mathfrak{A}$

Def: ω is a g.s. if $\omega(A^*[H,A]) \ge 0$ for any local operator A

Def: a unique g.s. ω is accompanied by a nonzero gap if there is $\gamma > 0$ and $\omega(A^*[H,A]) \ge \gamma\,\omega(A^*A)$ for any A s.t. $\omega(A) = 0$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant spin chains



$$\boldsymbol{S}_j = (S_j^{\mathrm{x}}, S_j^{\mathrm{y}}, S_j^{\mathrm{z}}) \quad j \in \mathbb{Z} \qquad [S_j^{\mathrm{x}}, S_k^{\mathrm{y}}] = i\delta_{j,k} S_j^{\mathrm{z}}, \ldots$$

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{e}, \mathrm{x}, \mathrm{y}, \mathrm{z}\} \quad u_{\mathrm{e}}^{(j)} = I \quad u_g^{(j)} = \exp[-i\pi S_j^g]$$

$$\mathrm{ind}_j = \begin{cases} 0 & \text{the spin at } j \text{ is an integer} \\ 1 & \text{the spin at } j \text{ is a half-odd-integer} \end{cases}$$

π-rotations about the three axes

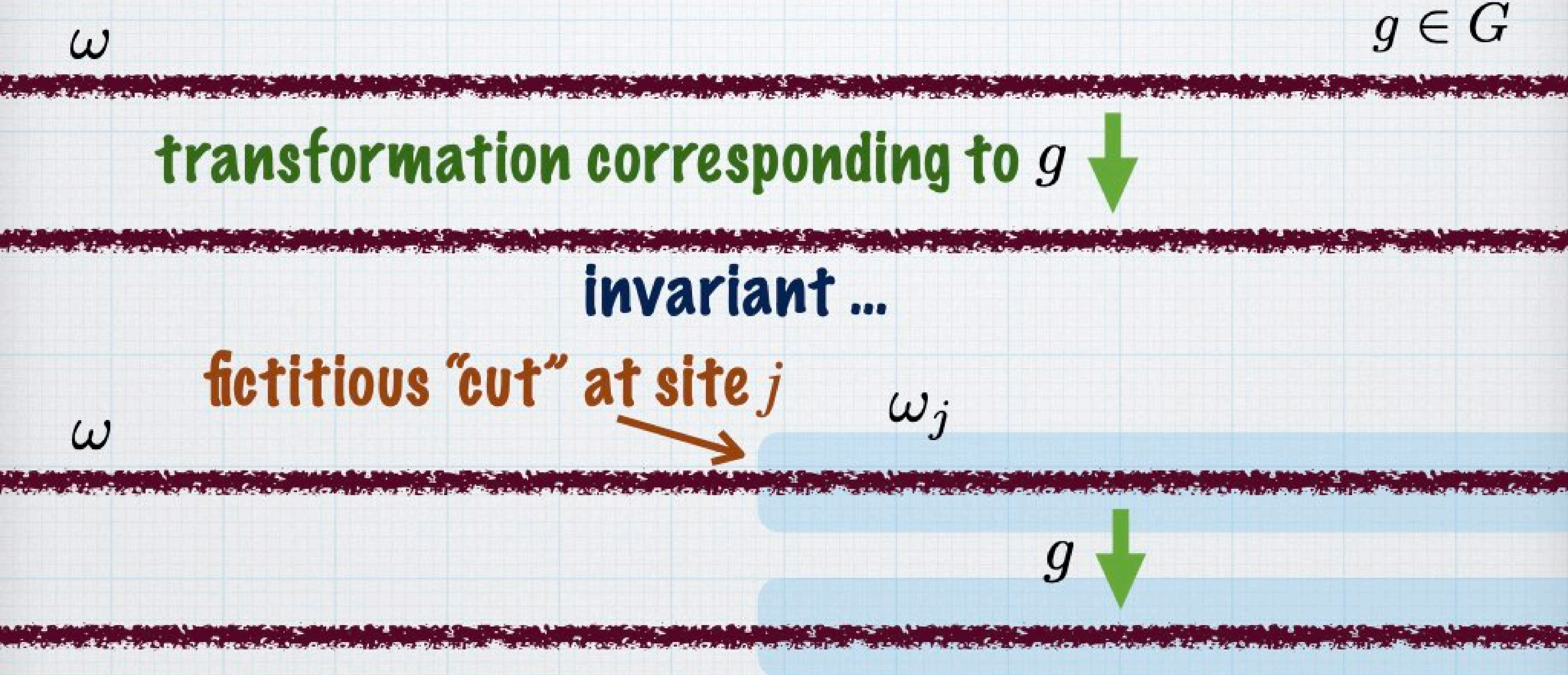
Ξ_{x}	$S_j^{\mathrm{x}} \to S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to -S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$
Ξ_{y}	$S_j^{\mathrm{x}} \to -S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$
Ξ_{z}	$S_j^{\mathrm{x}} \to -S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to -S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to S_j^{\mathrm{z}}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant models with a unique gapped g.s. $(S = 1)$

$$H_{\mathrm{AKLT}} = \sum_j \{\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \tfrac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\} \qquad H_{\mathrm{trivial}} = \sum_j (S_j^{\mathrm{z}})^2$$

index for a unique gapped ground state

We want define an index for a G-invariant unique gapped ground state ω on infinite chain



the state restricted on the half-infinite chain may exhibit nontrivial transformation property → index Ind_j

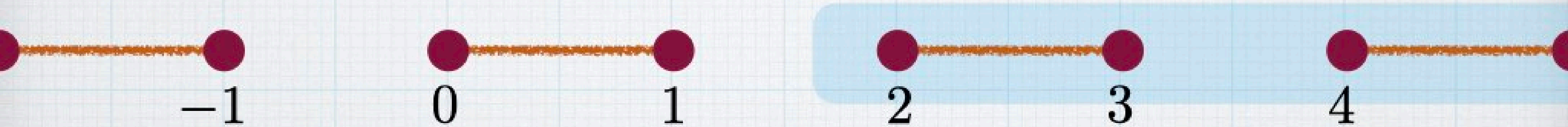
Easy example dimerized state

$\mathbb{Z}_2 \times \mathbb{Z}_2$ **invariant** $H = \sum_{j\in\mathbb{Z}} \boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j+1}$ **with** $S = \frac{1}{2}$

unique gapped g.s. $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} \frac{|\uparrow\rangle_{2j}|\downarrow\rangle_{2j+1} - |\downarrow\rangle_{2j}|\uparrow\rangle_{2j+1}}{\sqrt{2}}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ **invariant**

invariant



Ind_2 **should be** 0

free $S = 1/2$ **spin at the edge**

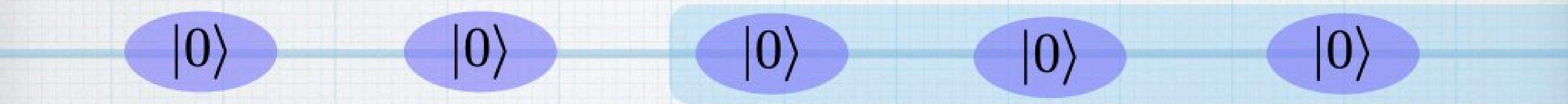
Ind_1 **should be** 1

Examples in $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1$ chains

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

$\mathfrak{h}_j = \mathrm{span}(|0\rangle_j, |+\rangle_j, |-\rangle_j)$

$H = \sum_{j\in\mathbb{Z}} (S_j^{\mathrm{z}})^2$ **unique gapped g.s.** $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} |0\rangle_j$

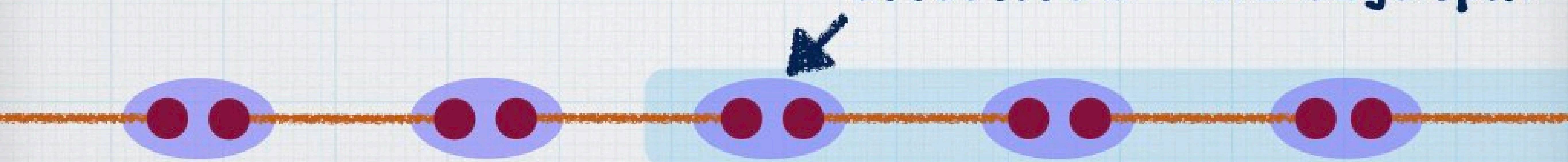


Ind_j **should be** 0

AKLT model $H = \sum_{j\in\mathbb{Z}} \{ \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 \}$

unique gapped g.s. = VBS state

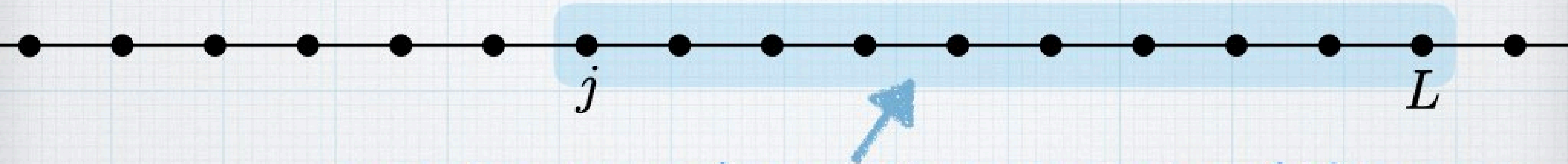
effective $S = 1/2$ **edge spin**



Ind_j **should be** 1

How should we define such an index for a unique gapped g.s.?

a large but finite system

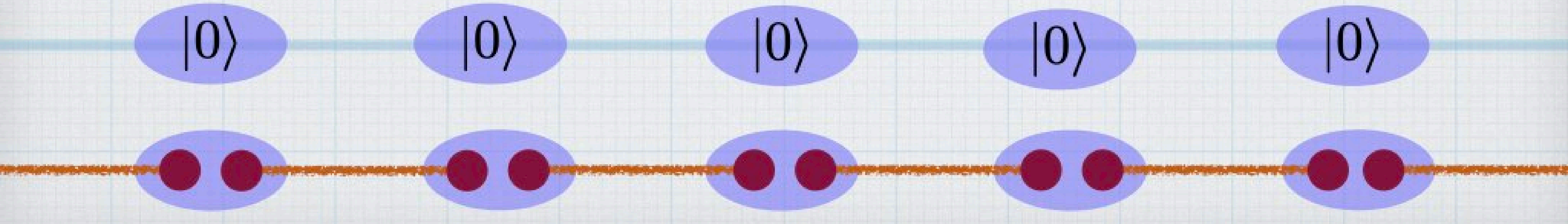


examine transformation property of this part

$\mathrm{Ind}_j = \sum_{k=j}^{L} \mathrm{ind_k}$ $\quad$ ind_k index for the single spin at site k

does not reflect the property of the g.s.!

we always have $\mathrm{Ind}_j = 0$ for $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric $S = 1$ chain



Index for matrix product states

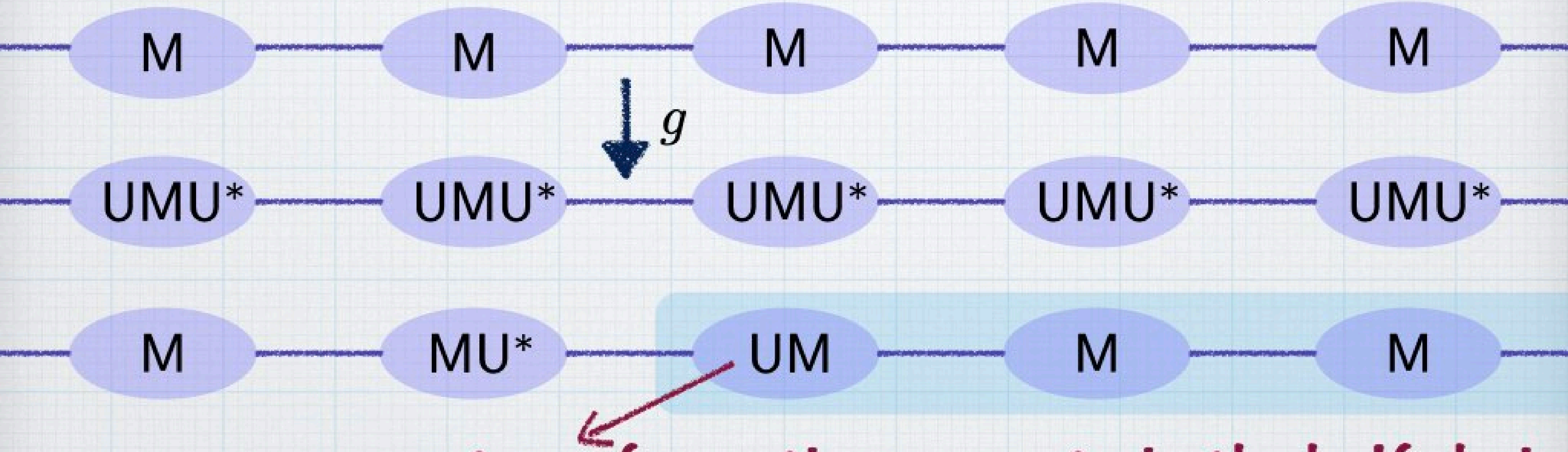
matrix product states (MPS) = finitely correlated states

Fannes, Nachtergaele, Werner 1989, 1992

G-invariant injective MPS

$$|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

matrices transform as $\mathsf{M}^{\sigma} \;\to\; \tilde{\mathsf{M}}^{\sigma} = \zeta_g\,\mathsf{U}_g\mathsf{M}^{\sigma}\mathsf{U}_g^{*}$



transformation property in the half chain

U_g proj. rep. of G ⟶ desirable index $\mathrm{Ind} \in \mathrm{H}^2(G, \mathrm{U}(1))$

Pollmann, Turner, Berg, Oshikawa 2010

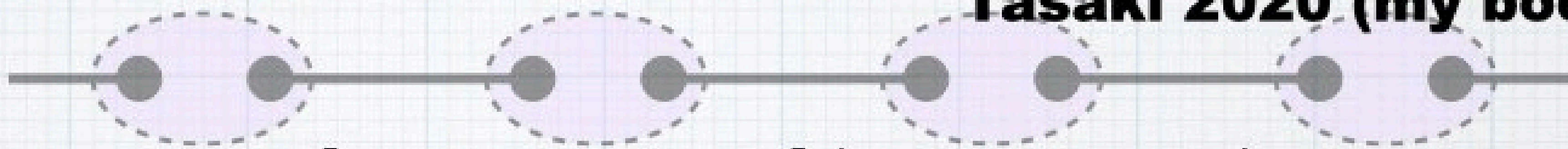
Perez-Garcia, Wolf, Sanz, Verstraete, and Cirac 2008

Fannes, Nachtergaele, Werner 199?

Matsui 2001

Index for the VBS state $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant state

Pollmann, Turner, Berg, Oshikawa 2010
Tasaki 2020 (my book)



$$|\mathrm{VBS}\rangle = \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

$$\mathsf{M}^{+} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \mathsf{M}^{0} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad \mathsf{M}^{-} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

π-rotation about the z-axis

$$u_{\mathrm{z}}^{(j)} = \exp[-i\pi S_j^{\mathrm{z}}] \qquad u_{\mathrm{z}}^{(j)}|\sigma\rangle_j = (-1)^{\sigma}|\sigma\rangle_j \quad \sigma = 0, \pm1$$

$$\Big(\bigotimes_{j=1}^{L} u_{\mathrm{z}}^{(j)}\Big)|\mathrm{VBS}\rangle =$$

$$= \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,(-1)^{\sum\sigma_j}\,|\sigma_1,\ldots,\sigma_L\rangle$$

$$= \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1}\ldots\tilde{\mathsf{M}}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

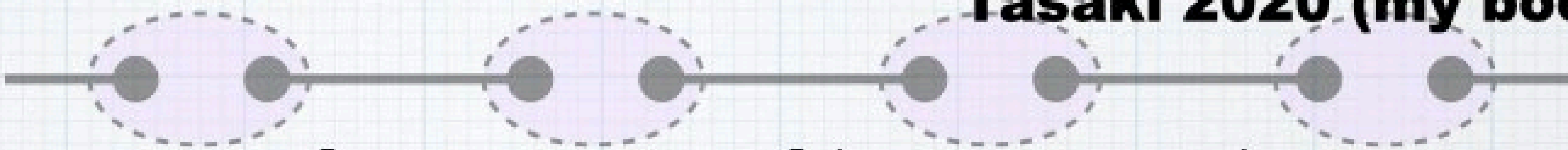
$$\tilde{\mathsf{M}}^{\sigma} = (-1)^{\sigma}\mathsf{M}^{\sigma} = -\mathsf{U}_{\mathrm{z}}\mathsf{M}^{\sigma}\mathsf{U}_{\mathrm{z}}^{*} \quad \sigma = 0, \pm1$$ **with** $\mathsf{U}_{\mathrm{z}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

similarly we get $\mathsf{U}_{\mathrm{x}} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \mathsf{U}_{\mathrm{y}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathsf{U}_{\mathrm{z}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

nontrivial projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad \mathrm{Ind} = 1$

Index for the VBS state $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant state

Pollmann, Turner, Berg, Oshikawa 2010
Tasaki 2020 (my book)



$$|\mathrm{VBS}\rangle = \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

$$\mathsf{M}^+ = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \mathsf{M}^0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad \mathsf{M}^- = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

π-rotation about the z-axis

$$u_{\mathrm{z}}^{(j)} = \exp[-i\pi S_j^{\mathrm{z}}] \qquad u_{\mathrm{z}}^{(j)}|\sigma\rangle_j = (-1)^\sigma|\sigma\rangle_j \quad \sigma = 0, \pm1$$

$$\Bigl(\bigotimes_{j=1}^{L} u_{\mathrm{z}}^{(j)}\Bigr)|\mathrm{VBS}\rangle =$$

$$= \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,(-1)^{\sum\sigma_j}\,|\sigma_1,\ldots,\sigma_L\rangle$$

$$= \sum_{\sigma_1,\ldots,\sigma_L=0,\pm1} \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1}\ldots\tilde{\mathsf{M}}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

$$\tilde{\mathsf{M}}^\sigma = (-1)^\sigma\mathsf{M}^\sigma = -\mathsf{U}_{\mathrm{z}}\ldots$$

π-rotation operators for $S = 1/2$

similarly we get $\mathsf{U}_{\mathrm{x}} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ $\mathsf{U}_{\mathrm{y}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\mathsf{U}_{\mathrm{z}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

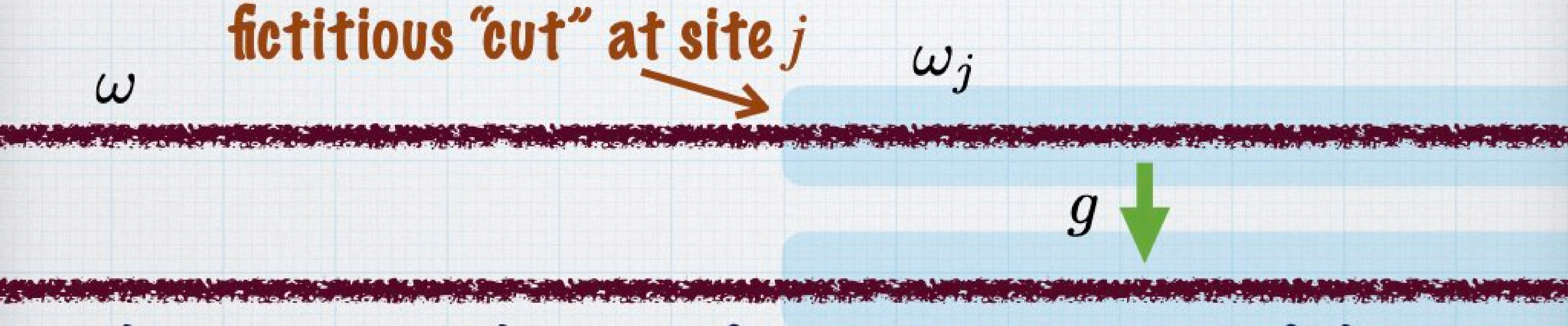
nontrivial projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\mathrm{Ind} = 1$

Ogata index

Ogata index for a general G-invariant unique gapped ground state

Ogata 2018 (Matsui 2001, 2013)

definition of index $\mathrm{Ind}_j^{\omega} \in \mathrm{H}^2(G, \mathrm{U}(1))$ for any G-invariant unique gapped g.s. (more generally, pure split state)



characterizes the transformation property of the g.s. restricted on the half-infinite chain

$$\pi_j(\mathfrak{A}_j) \subset \pi_j(\mathfrak{A}_j)'' \subset B(\mathcal{H}_j)$$

$$\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$$

operator algebraic formulation
representation on the type I factor

$\pi_j(\mathfrak{A}_j)$ —bicommutant→ $\pi_j(\mathfrak{A}_j)''$

representation of the C* algebra

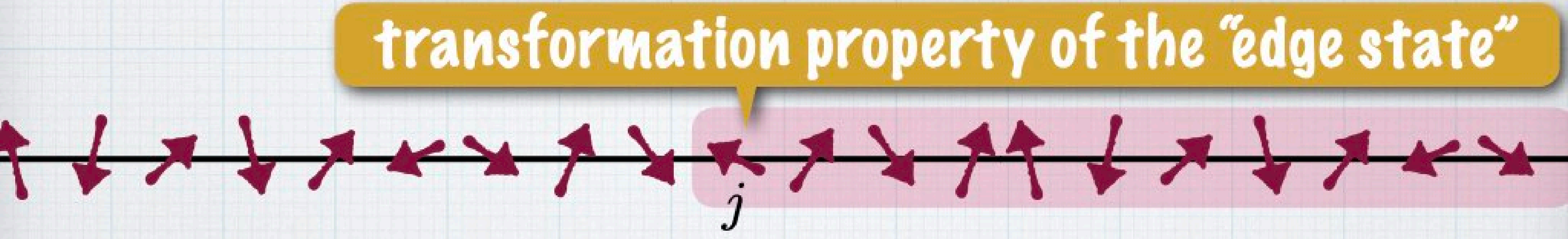
von Neum[ann] algebra

illustration by Chisato Naruse

basic properties of Ogata index

G-invariant unique gapped ground state ω site $j \in \mathbb{Z}$

- well defined index $\mathrm{Ind}_j^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$



- identical to the index of Pollmann, Turner, Berg, and Oshikawa for MPS (matrix product states)

Ogata 2018

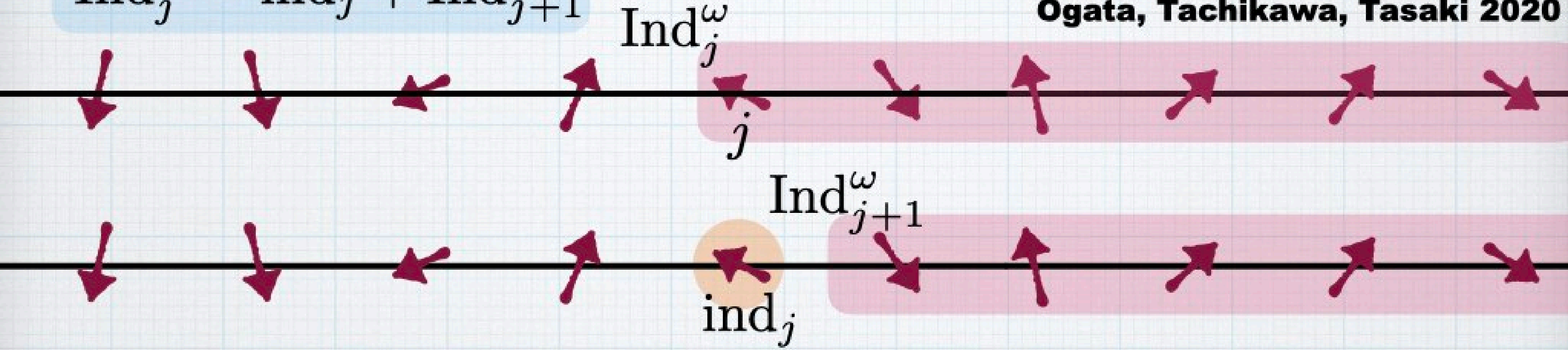
- invariant under smooth modification of G-invariant models with a unique gapped ground state

Ogata 2018

properties of Ogata index: additivity

$$\mathrm{Ind}^{\omega}_{j} = \mathrm{ind}_j + \mathrm{Ind}^{\omega}_{j+1}$$

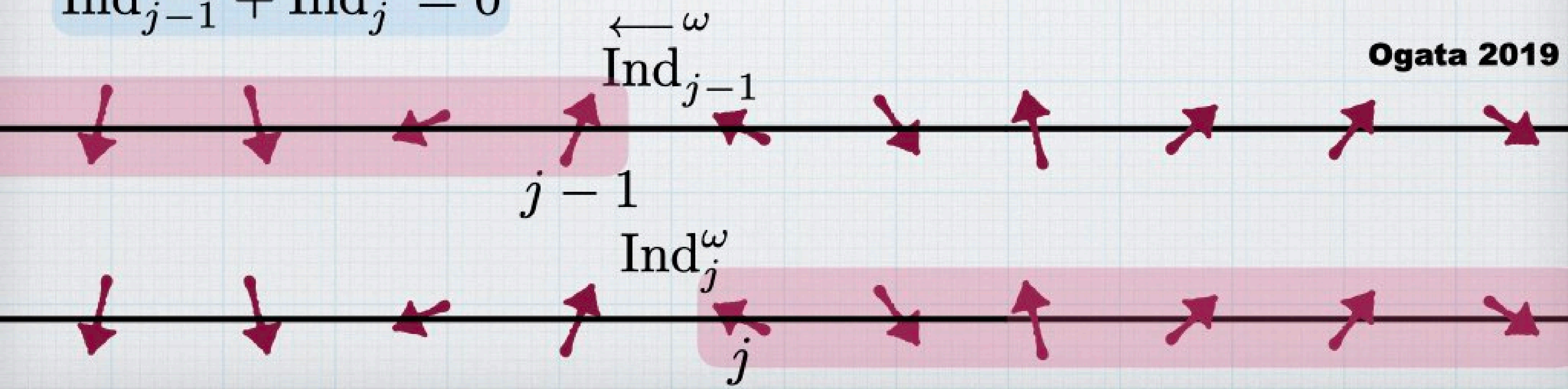
Ogata, Tachikawa, Tasaki 2020



transformation property of the ground state restricted to the left half-infinite chain

$$\overleftarrow{\mathrm{Ind}}^{\omega}_{j-1} + \mathrm{Ind}^{\omega}_{j} = 0$$

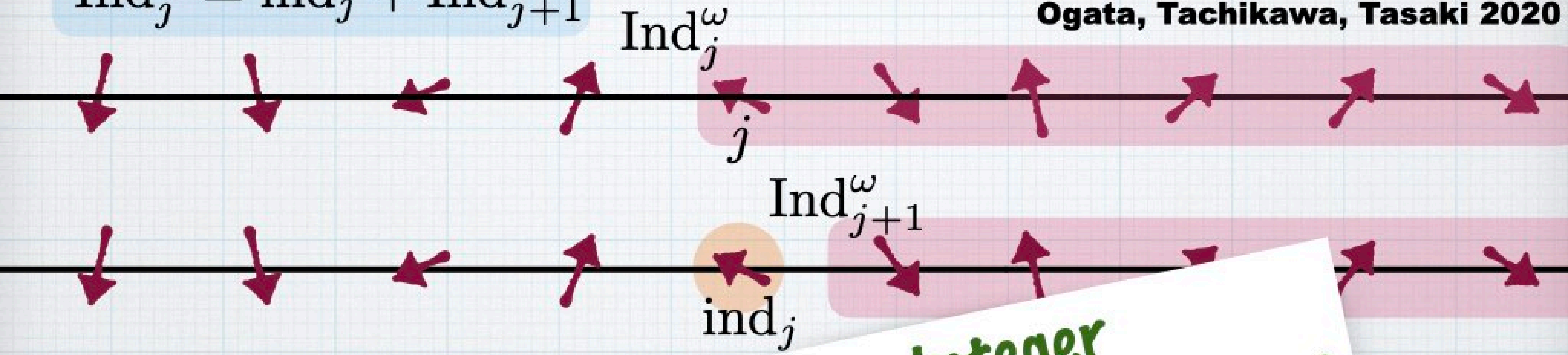
Ogata 2019



properties of Ogata index: additivity

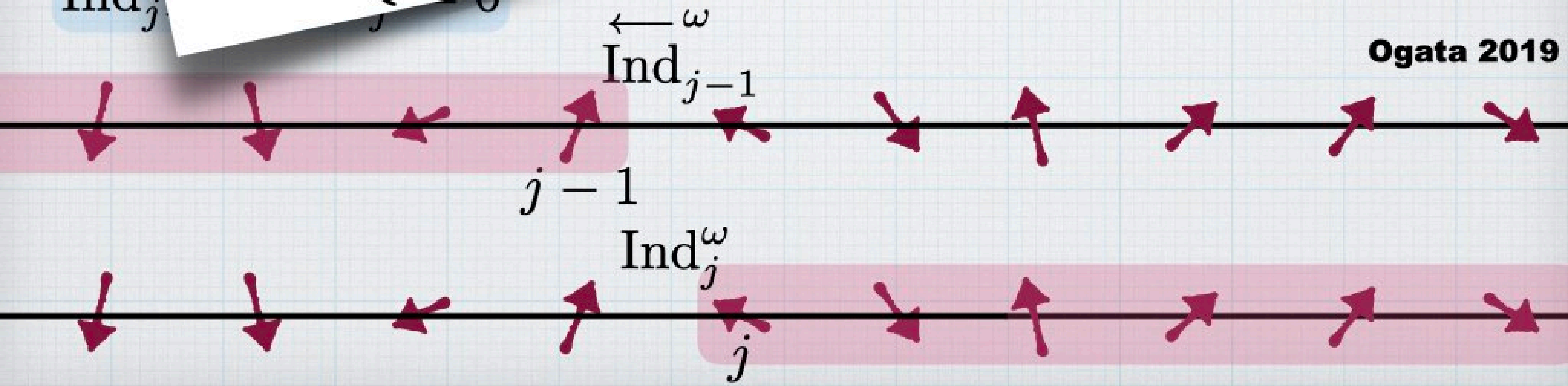
$$\mathrm{Ind}^{\omega}_{j} = \mathrm{ind}_j + \mathrm{Ind}^{\omega}_{j+1}$$

Ogata, Tachikawa, Tasaki 2020



$$\mathrm{ind}_j = \begin{cases} 0 & \text{the spin at } j \text{ is an integer} \\ 1 & \text{the spin at } j \text{ is a half-odd-integer} \end{cases}$$

Ogata 2019

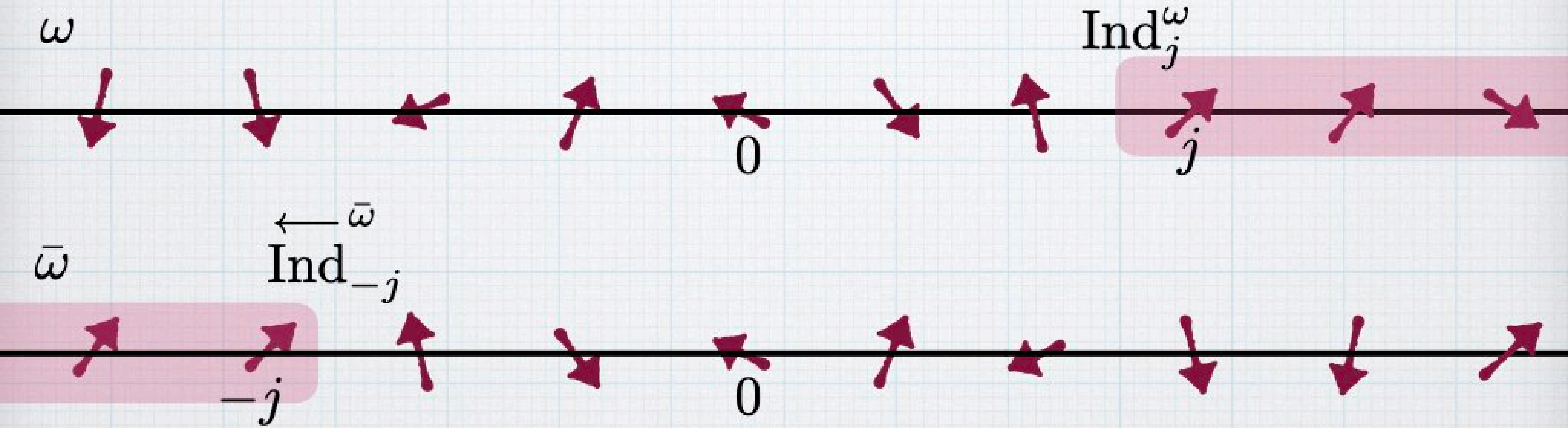


properties of Ogata index: inversion

spatial inversion

$$\mathrm{Ind}_j^{\bar{\omega}} = -\mathrm{Ind}_{1-j}^{\omega}$$

$$\omega \xrightarrow{\; j \to -j \;} \bar{\omega}$$



$$\mathrm{Ind}_j^{\omega} = \overleftarrow{\mathrm{Ind}}_{-j}^{\bar{\omega}} = -\mathrm{Ind}_{-j+1}^{\bar{\omega}}$$

trivial

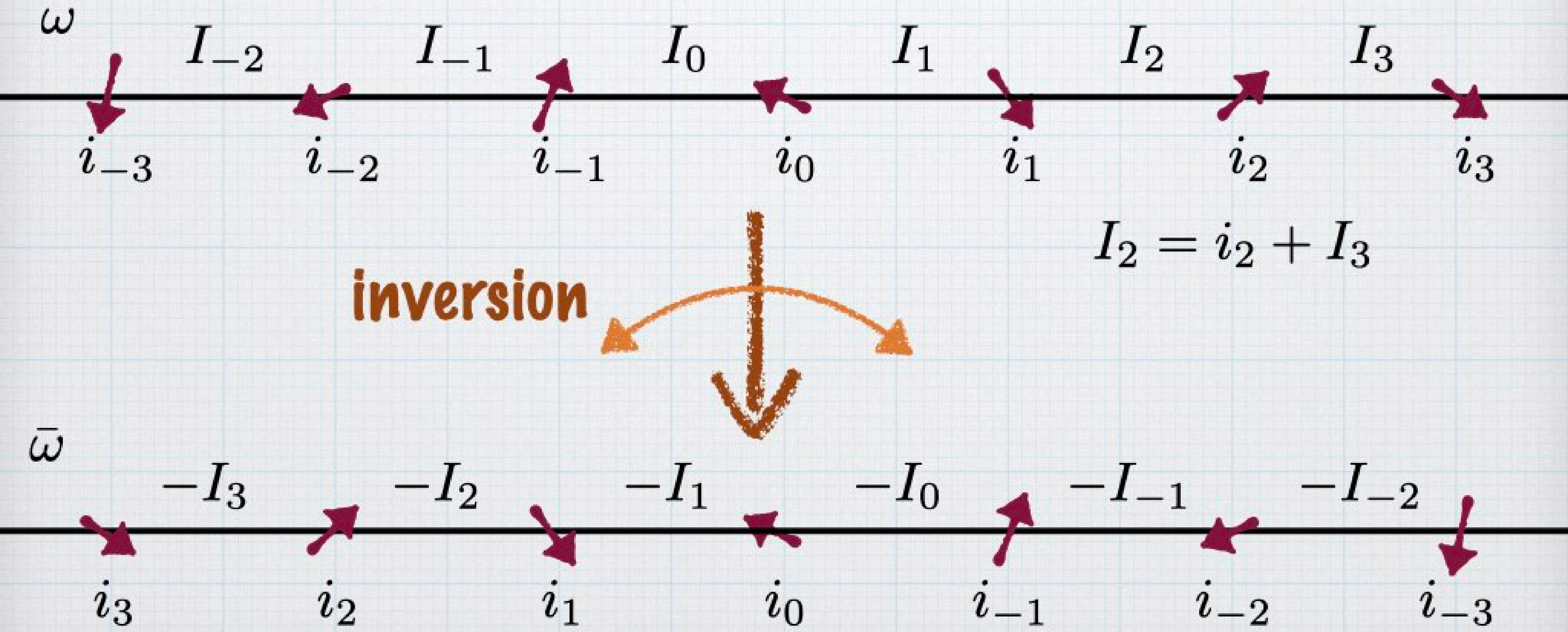
additivity

graphic notation and summary of the properties

$i_j = \mathrm{ind}_j \in \mathrm{H}^2(G, U(1))$ index for spin at j

$I_j = \mathrm{Ind}^{\omega}_j \in \mathrm{H}^2(G, U(1))$

Ogata index for ω restricted to $\{j, j+1, \ldots\}$



Classification of symmetry protected topological (SPT) phases

SPT phases of unique gapped g.s.

H_0, H_1 G-invariant Hamiltonians of the same spin chain with unique gapped ground states ω_0, ω_1

definition

the two models belong to the same SPT phase protected by the symmetry G iff there exists a family of G-invariant Hamiltonians H_s with $0 \leq s \leq 1$ such that

H_s has a unique gapped g.s. ω_s for each s

ω_s depends smoothly on s

$\mathrm{Ind}_j^{\omega} \in \mathrm{H}^2(G, \mathrm{U}(1))$ index for a unique gapped g.s. ω

essential theorem **Ogata 2018**

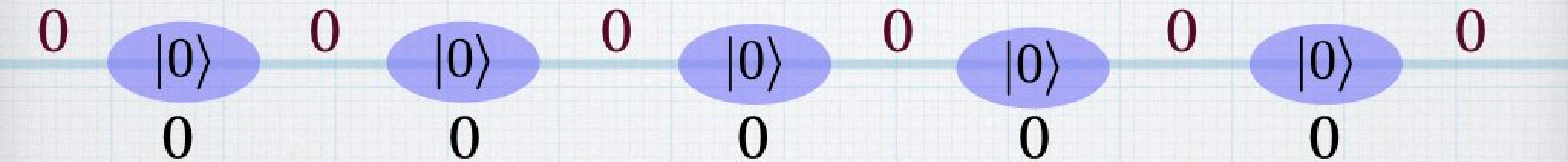
Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then the two models belong to different SPT phases protected by the symmetry G

Ind_0^{ω} determines Ind_j^{ω} for all j because $\mathrm{Ind}_j^{\omega} = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^{\omega}$

example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1$ chains

Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then the two models belong to different SPT phases protected by the symmetry G

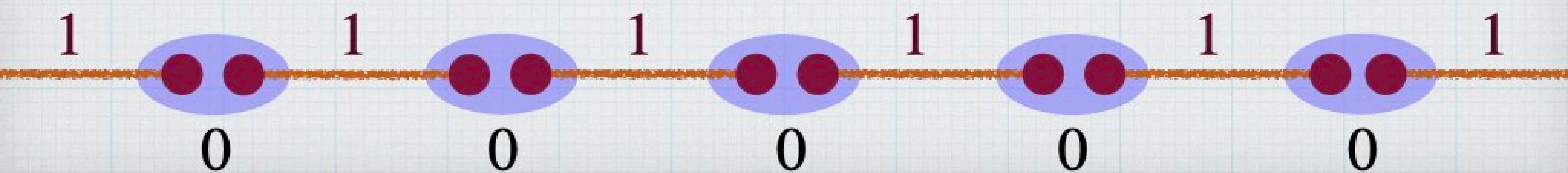
$H = \sum_{j\in\mathbb{Z}} (S_j^{\mathrm{z}})^2$ unique gapped g.s. $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} |0\rangle_j$



the two models (ground states) belong to distinct SPT phases protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

AKLT model $H = \sum_{j\in\mathbb{Z}} \{ \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3} (\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 \}$

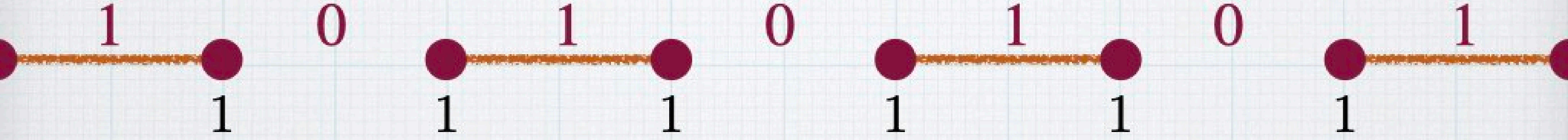
unique gapped g.s. = VBS state



example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1/2$ chains

Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then the two models belong to different SPT phases protected by the symmetry G

$$H_0 = \sum_{j\in\mathbb{Z}} \boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j+1} \qquad \Phi_0 = \bigotimes_{j\in\mathbb{Z}} \frac{|\uparrow\rangle_{2j}|\downarrow\rangle_{2j+1} - |\downarrow\rangle_{2j}|\uparrow\rangle_{2j+1}}{\sqrt{2}}$$



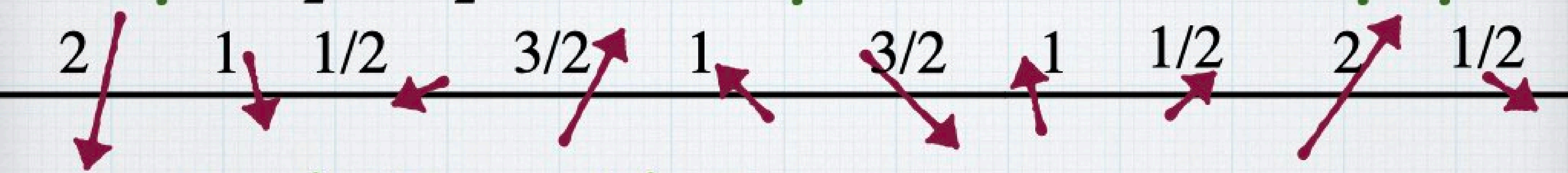
the two models (ground states) belong to distinct SPT phases protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

$$H_1 = \sum_{j\in\mathbb{Z}} \boldsymbol{S}_{2j-1} \cdot \boldsymbol{S}_{2j} \qquad \Phi_1 = \bigotimes_{j\in\mathbb{Z}} \frac{|\uparrow\rangle_{2j-1}|\downarrow\rangle_{2j} - |\downarrow\rangle_{2j-1}|\uparrow\rangle_{2j}}{\sqrt{2}}$$

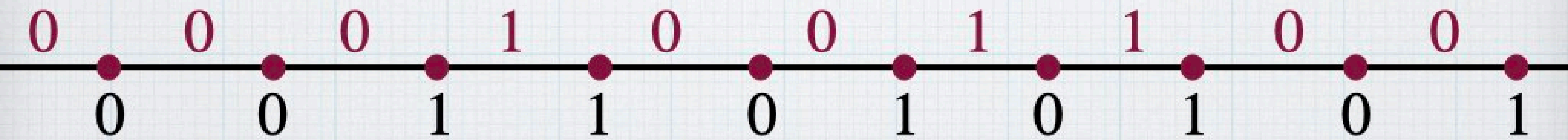
general classification by Ogata

for any G-invariant (not necessarily translation invariant) quantum spin chains, SPT phases protected by G are classified by a single index $\mathrm{Ind}_0^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$

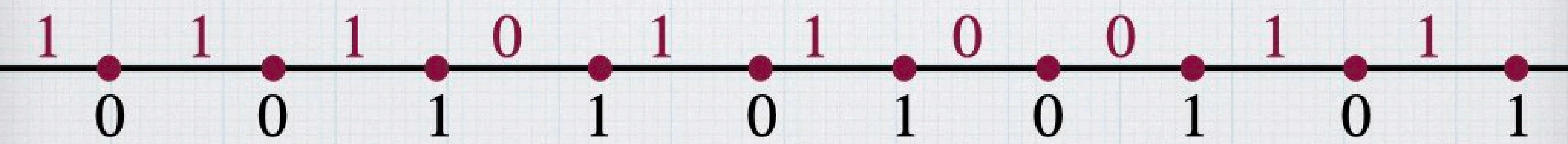
example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant spin chain with site-dep. spins



Ind_j^ω at each site can either be



or



Ind_0^ω determines Ind_j^ω for all j because $\mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$

general classification by Ogata

for any G-invariant (not necessarily translation invariant) quantum spin chains, SPT phases protected by G are classified by a single index $\mathrm{Ind}_0^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$

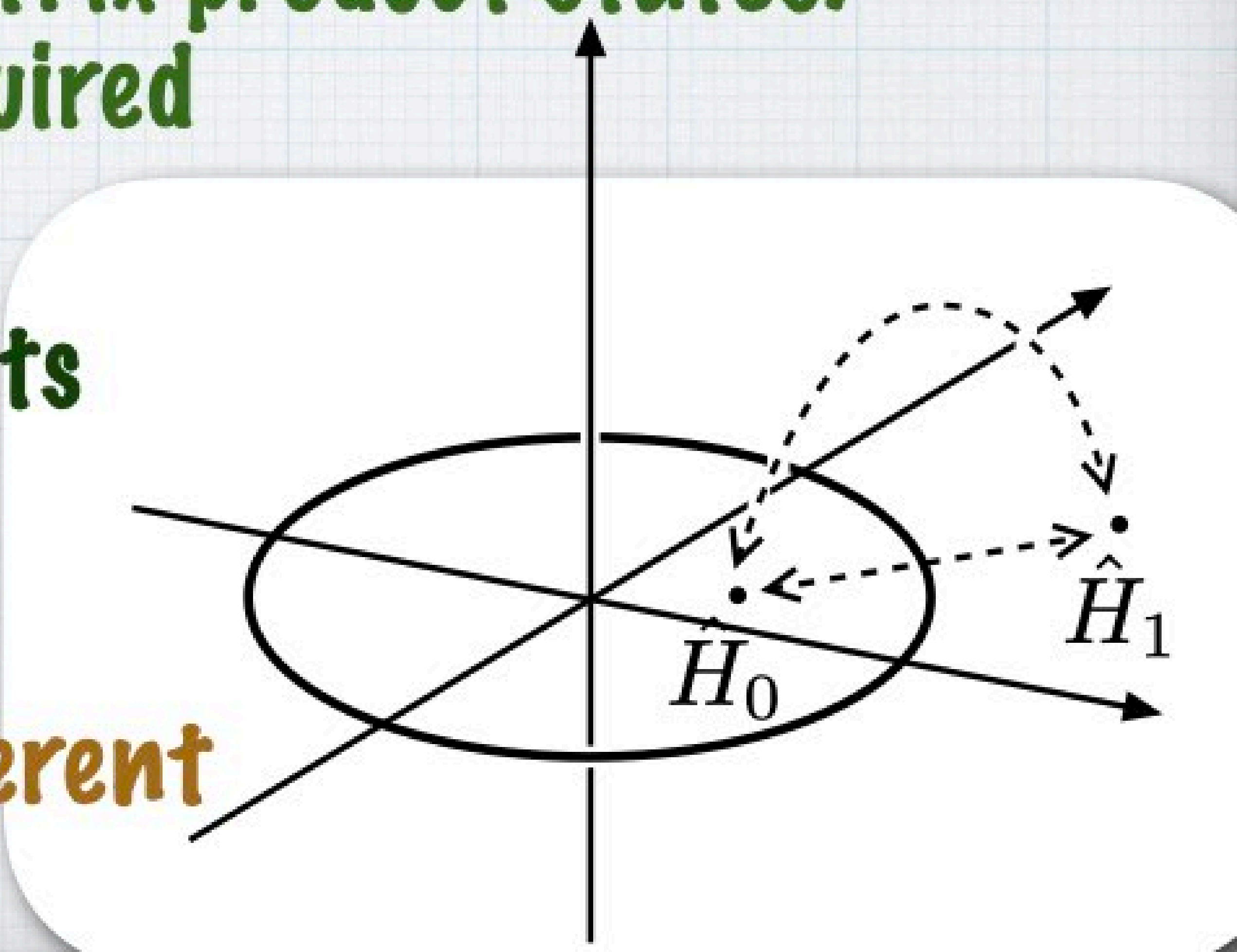
a strict extension of the known classifications

Pollmann, Turner, Berg, Oshikawa 2010

works for injective MPS (matrix product states)
translation invariance is required

Chen, Gu, Wen 2011

classification of RG fixed points
some uniformity is assumed



Remark: it is possible that different "phases" have the same index

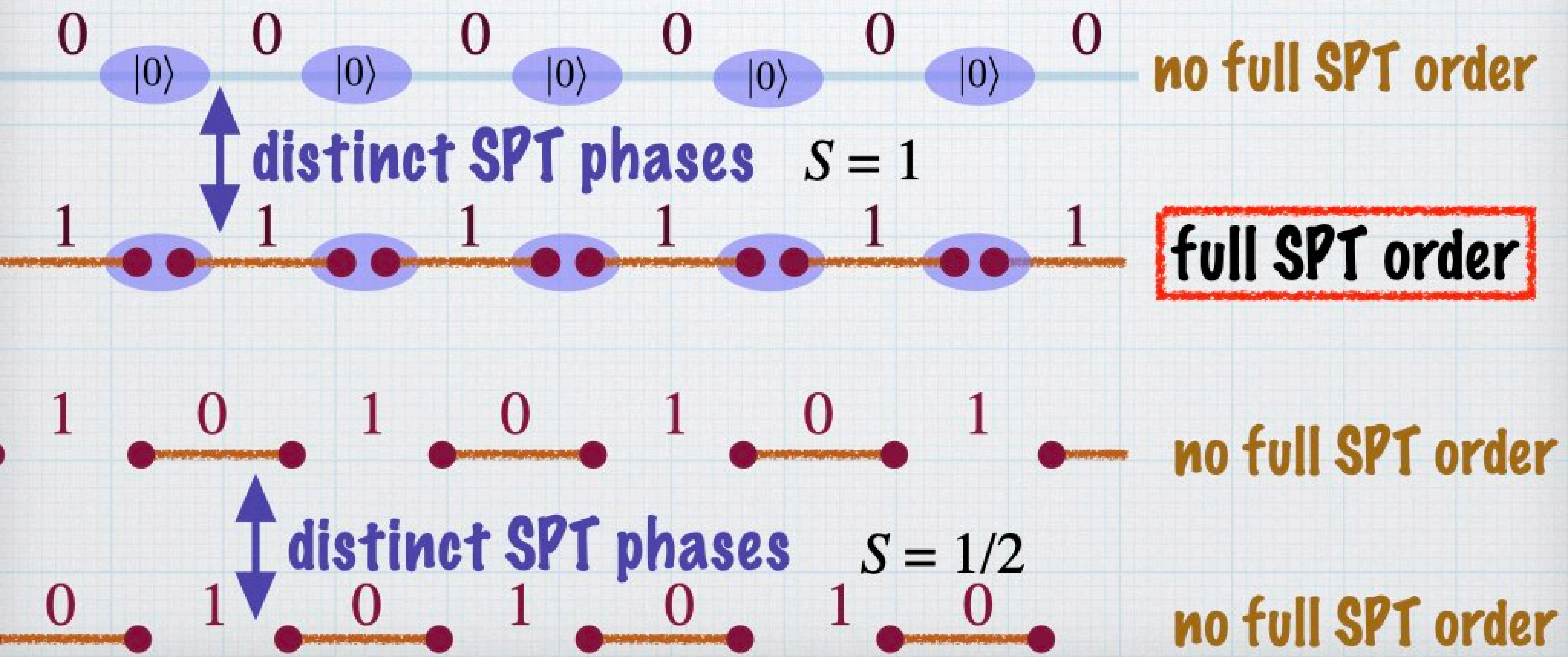
Full SPT order

the concept of full SPT (symmetry protected topological) order

definition

Ogata, Tasaki (in preparation)

G-invariant unique gapped ground state ω has full SPT order if and only if $\mathrm{Ind}_j^{\omega} \neq 0$ for any $j \in \mathbb{Z}$



full SPT order and entanglement

definition

G-invariant unique gapped ground state ω has full SPT order if and only if $\mathrm{Ind}_j^\omega \neq 0$ for any $j \in \mathbb{Z}$

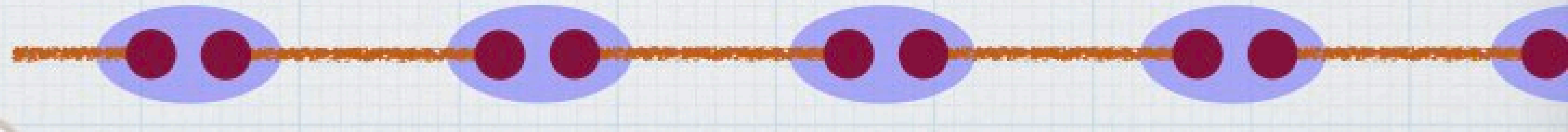
S_j^ω entanglement entropy between $\{\ldots, j-1\}$ and $\{j, \ldots\}$

Ogata 2018

Theorem: If ω has full SPT order then $S_j^\omega \geq \log 2$ for any $j \in \mathbb{Z}$

entanglement enforced by symmetry!

Pollmann, Turner, Berg, Oshikawa 2010



https://www.irasutoya.com/2014/02/blog-post_2233.html

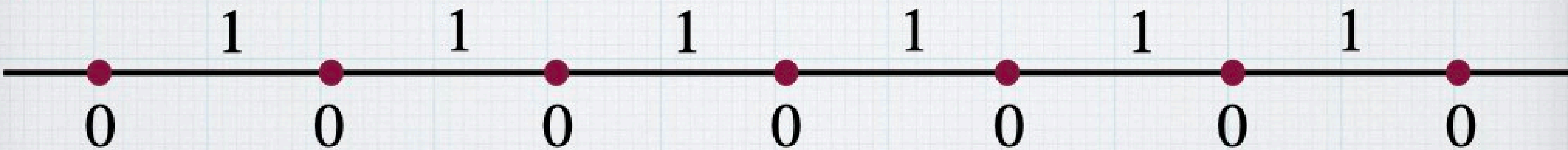
a necessary condition for full SPT order

definition

G-invariant unique gapped ground state ω has full SPT order if and only if $\mathrm{Ind}_j^\omega \neq 0$ for any $j \in \mathbb{Z}$

Ogata, Tasaki (in preparation

Theorem: If $\mathrm{H}^2(G, \mathrm{U}(1)) = \{0, 1\}$ then full SPT order is possible only for a spin chain with $\mathrm{ind}_j = 0$ for all $j \in \mathbb{Z}$



$$\mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$$

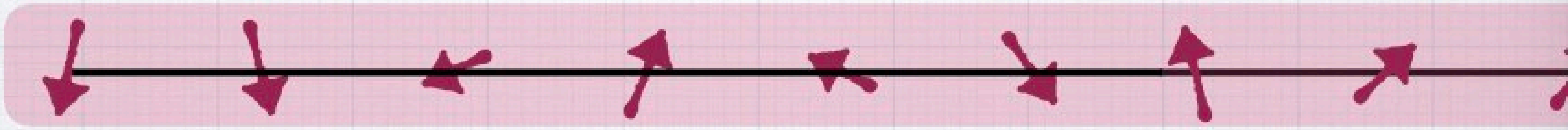
We find Haldane phase (protected by on-site symmetry) only for integer spins!

Remark: In a translationally invariant model with ANY symmetry, full SPT order is possible only when $\mathrm{ind}_j = 0$ for all $j \in \mathbb{Z}$ (Lieb-Schultz-Mattis type theorem)

half-infinite chains

spin system on the half-infinite chain $\mathbb{Z}_+ = \{0, 1, \ldots\}$

G-invariant unique gapped ground state ω



$\mathrm{Ind}_0^\omega = 0 \qquad \mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$

$$\mathrm{Ind}_j^\omega = -\sum_{k=0}^{j-1} \mathrm{ind}_k \quad j = 1, 2, \ldots$$

we know all the Ogata indices completely!

model with $\mathrm{H}^2(G, \mathrm{U}(1)) = \{0, 1\}$

$\mathrm{ind}_j = 0$ for all $j = 0, 1, \ldots$ → $\mathrm{Ind}_j^\omega = 0$ for all $j = 0, 1, \ldots$

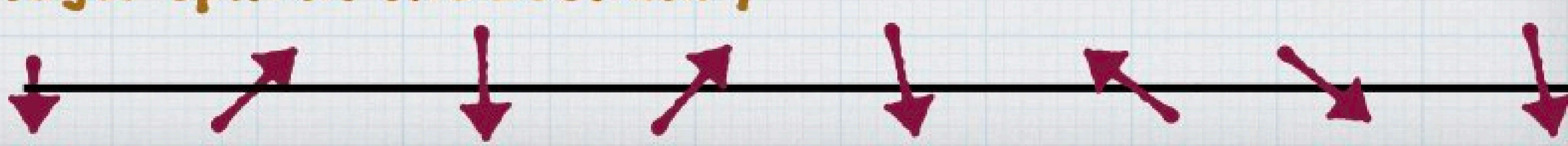
integer spin chain

full SPT order

$\mathrm{ind}_0 = 1$ and $\mathrm{ind}_j = 0$ for all $j = 1, 2, \ldots$

→ $\mathrm{Ind}_j^\omega = 1$ for all $j = 1, 2, \ldots$

integer spin chain with
a half-odd-integer spin at the boundary



enforcement of full SPT order by symmetry

SPT-LSM theorem

Jiang, Cheng, Qi, Lu 2019

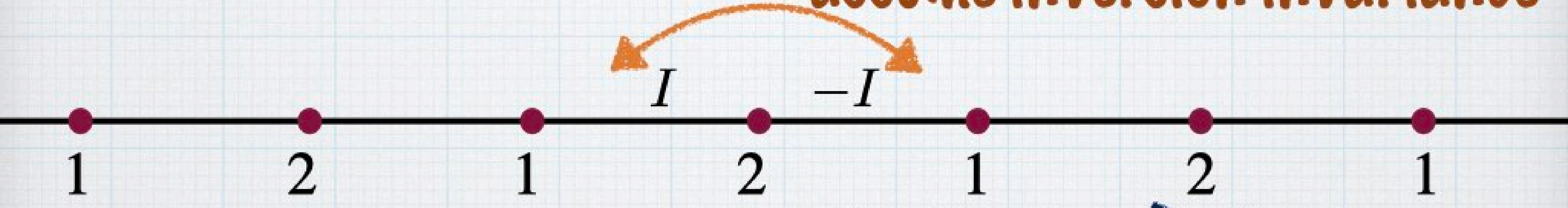
Haruki Watanabe, private communication

Ogata, Tasaki (in preparation)

simplest example: unique gaped ground state invariant under $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathrm{SU}(3)$

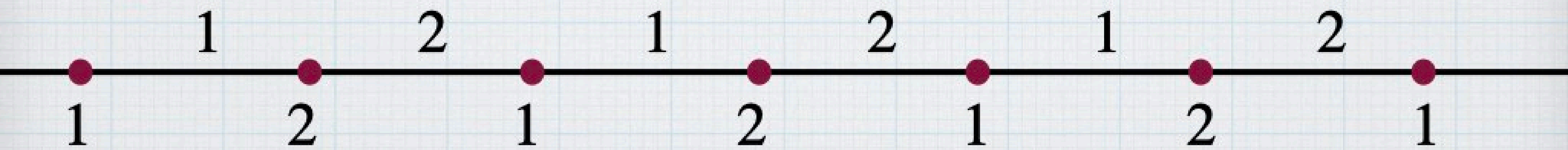
$\mathrm{H}^2(G, \mathrm{U}(1)) = \mathbb{Z}_3 = \{0, 1, 2\}$

assume inversion invariance



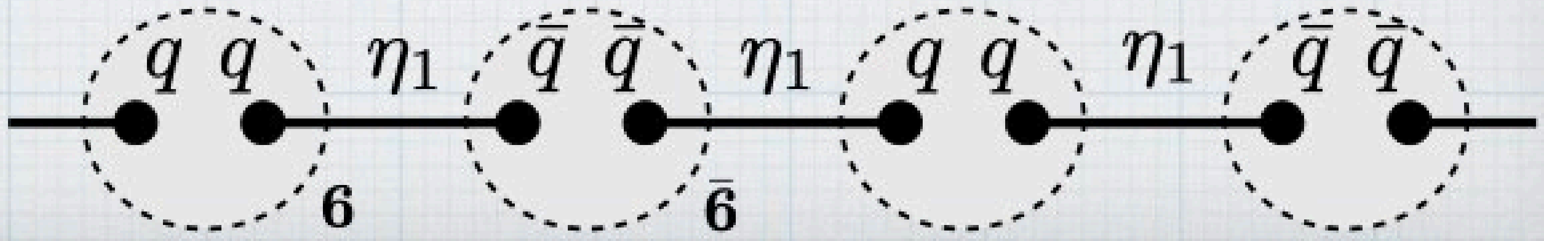
$I = 2 - I \longrightarrow I = 1$

the ground state must have full SPT order



$\mathrm{SU}(3)$ AKLT state

$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$ $\quad$ $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{6}} \oplus \mathbf{3}$



enforcement of full SPT order by symmetry

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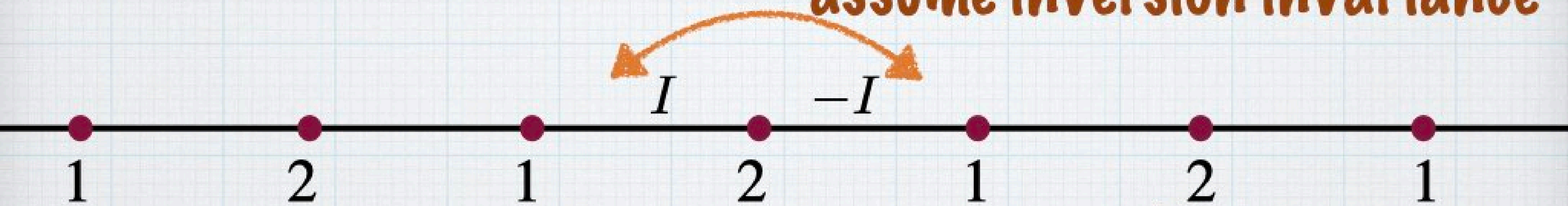


$I = 2 - I \longrightarrow I = 1$

the ground state must have full SPT order

something nontrivial is guaranteed to take place in this "spin" system

non-unique g.s., gapless g.s., or full SPT order

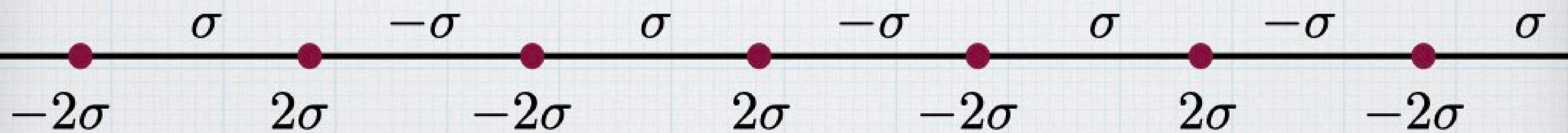
enforcement of full SPT order by symmetry

Jiang, Cheng, Qi, Lu 2019

Haruki Watanabe, private communication

Ogata, Tasaki (in preparation)

Theorem: Let $\sigma \in \mathrm{H}^2(G, \mathrm{U}(1))$ be such that $2\sigma \neq 0$, and consider a "spin chain" with $\mathrm{ind}_j = 2\sigma$ for even j and $\mathrm{ind}_j = -2\sigma$ for odd j, and assume that the model is invariant under inversion about the origin. Then the ground state of the model has full SPT order if it is unique and gapped.



inversion symmetry can be replaced by symmetry under {translation + $\mathrm{ind} \to -\mathrm{ind}$}

Summary of parts 1 and 2

- ☑ Haldane's discovery in 1983 about unique gapped g.s. in spin chains has lead to the notion of symmetry protected topological (SPT) phases
- ☑ Ogata index is defined for any G-invariant unique gapped g.s., and characterizes the transformation property of the g.s. restricted on the half-infinite chain
- ☑ Ogata index provides us with a classification of SPT phases in quantum spin chains protected by symmetry G that strictly extends the classifications of Pollmann-Turner-Berg-Oshikawa and Chen-Gu-Wen
- ☑ the classification suggests a natural notion of full SPT order