Ogata index for quantum spin chains and its applications

Hal Tasaki

Y. Ogata, CMP 2019 and more
Y. Ogata and H. Tasaki, CMP 2019
Y. Ogata, Y. Tachikawa, and H. Tasaki, arXiv:2004.06458
Y. Ogata and H. Tasaki, in preparation

Index for projective representation of symmetry group

Index for a unique gapped ground state

Application 1: LSM type theorems

Application 2: classification of SPT

Definition of Ogata index

Representation and projective representation of a group

G a finite group (for the symmetry of the model)

representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = U_{gh}$

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

$$\varphi(g,h) \in \mathrm{U}(1) := \{z \in \mathbb{C} \,|\, |z| = 1\}$$

associativity $U_f(U_g U_h) = (U_f U_g) U_h$ implies

$$\frac{\varphi(g,h)\,\varphi(f,gh)}{\varphi(f,g)\,\varphi(fg,h)} = 1 \quad \text{for any } f,g,h \in G$$

φ is a 2-cocycle

$\mathrm{Z}^2(G,\mathrm{U}(1))$ the set of all 2-cocycles $\varphi : G \times G \to \mathrm{U}(1)$

Remark: we can treat proj. reps. with antiuniatary operators

Index for a projective representation

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

equivalent projective representation

$$U'_g = \psi(g)\, U_g \qquad \psi(g) \in \mathrm{U}(1) := \{z \in \mathbb{C} \,|\, |z| = 1\}$$

$$U'_g U'_h = \varphi'(g,h)\, U'_{gh} \qquad \varphi'(g,h) = \frac{\psi(g)\psi(h)}{\psi(gh)}\varphi(g,h)$$

$\varphi \sim \varphi'$ iff $\varphi'(g,h) = \dfrac{\psi(g)\psi(h)}{\psi(gh)}\varphi(g,h)$ with some $\psi(g)$

second group cohomology $\mathrm{H}^2(G, \mathrm{U}(1)) = \mathrm{Z}^2(G, \mathrm{U}(1))/\sim$

abelian group

$\mathrm{H}^2(G, \mathrm{U}(1)) \ni \mathrm{ind}$ characterizes an equivalence class of the projective representations of G

Index for a projective representation

projective representation

unitary U_g (with $g \in G$) s.t. $U_e = I$ and $U_g U_h = \varphi(g,h)\, U_{gh}$

$\mathrm{H}^2(G, \mathrm{U}(1)) \ni \mathrm{ind}$ characterizes an equivalence class of the projective representations of G

the indices can be added!

two projective representations

$$u_g^{(1)} \qquad u_g^{(1)} u_h^{(1)} = \varphi_1(g,h)\, u_{gh}^{(1)} \qquad \mathrm{ind}_1$$

$$u_g^{(2)} \qquad u_g^{(2)} u_h^{(2)} = \varphi_2(g,h)\, u_{gh}^{(2)} \qquad \mathrm{ind}_2$$

$U_g = u_g^{(1)} \otimes u_g^{(2)}$ proj. rep with $\varphi(g,h) = \varphi_1(g,h)\, \varphi_2(g,h)$

$$\mathrm{ind} = \mathrm{ind}_1 + \mathrm{ind}_2$$

Important example $\mathbb{Z}_2 \times \mathbb{Z}_2$

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{e}, \mathrm{x}, \mathrm{y}, \mathrm{z}\}$

$\mathrm{x}^2 = \mathrm{y}^2 = \mathrm{z}^2 = \mathrm{e} \quad \mathrm{xy} = \mathrm{yx} = \mathrm{z} \quad \mathrm{yz} = \mathrm{zy} = \mathrm{x} \quad \mathrm{zx} = \mathrm{xz} = \mathrm{y}$

$\mathrm{H}^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2 = \{0, 1\} \ni \mathrm{ind}$

proj. rep. on a single spin $\boldsymbol{S} = (S^{\mathrm{x}}, S^{\mathrm{y}}, S^{\mathrm{z}}) \qquad \boldsymbol{S}^2 = S(S+1)$

$U_{\mathrm{e}} = I \qquad U_g = \exp[-i\pi S^g] \qquad g \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$

$U_{\mathrm{x}} U_{\mathrm{y}} = U_{\mathrm{z}} \qquad U_{\mathrm{y}} U_{\mathrm{z}} = U_{\mathrm{x}} \qquad U_{\mathrm{z}} U_{\mathrm{x}} = U_{\mathrm{y}}$

integer spin $(S = 1, 2, \ldots)$ **genuine representation**

$(U_g)^2 = I \qquad U_g U_h = U_h U_g$

half-odd-integer spin $(S = \frac{1}{2}, \frac{3}{2}, \ldots)$

$(U_g)^2 = -I \qquad U_g U_h = -U_h U_g$

$g, h \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$
$g \neq h$

nontrivial proj. rep.

$$\mathrm{ind} = \begin{cases} 0 & S = 1, 2, \ldots \quad \textbf{trivial} \\ 1 & S = \frac{1}{2}, \frac{3}{2}, \ldots \quad \textbf{nontrivial} \end{cases}$$

Index for projective representation of symmetry group

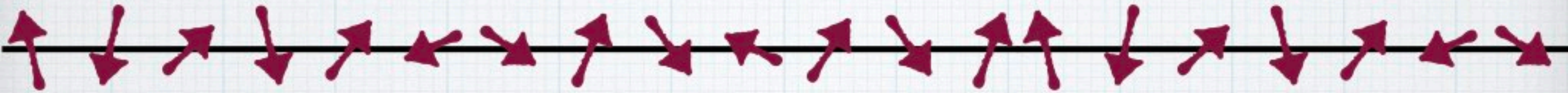
$\boxed{\text{Index for a unique gapped ground state}}$

Application 1: LSM type theorems

Application 2: classification of SPT

Definition of Ogata index

General quantum spin chain



$\mathfrak{h}_j$ Hilbert space at site $j \in \mathbb{Z}$ $\dim(\mathfrak{h}_j) \le d_0$

G finite group that describes the global on-site symmetry

$u_g^{(j)}$ unitary on $\mathfrak{h}_j$

projective representation with index $\mathrm{ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

transformation of operator A by $g \in G$

$$\Xi_g(A) = \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right) A \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right)^*$$

G-invariant short ranged Hamiltonian $H = \sum_{j\in\mathbb{Z}} h_j$

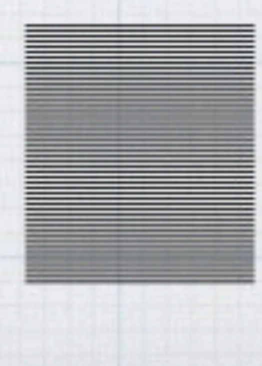
$h_j =$

"unique gapped ground state"

$\Xi_g(h_j) = h_j$ for any $j \in \mathbb{Z}$ and $g \in G$

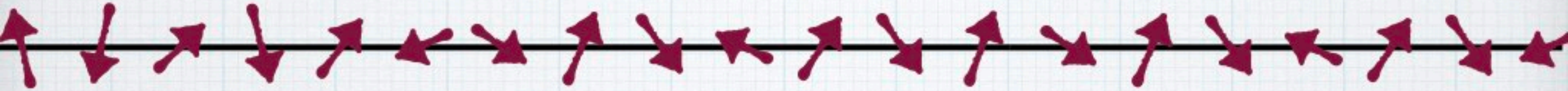
assume the model has a unique ground sate ω

accompanied by a nonzero energy gap



$\Delta E = O(1)$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant spin chains



$$\boldsymbol{S}_j = (S_j^{\mathrm{x}}, S_j^{\mathrm{y}}, S_j^{\mathrm{z}}) \quad j \in \mathbb{Z} \qquad\qquad [S_j^{\mathrm{x}}, S_k^{\mathrm{y}}] = i\delta_{j,k} S_j^{\mathrm{z}}, \ldots$$

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{e}, \mathrm{x}, \mathrm{y}, \mathrm{z}\} \quad u_{\mathrm{e}}^{(j)} = I \quad u_g^{(j)} = \exp[-i\pi S_j^g]$$

π-rotations about the three axes

Ξ_{x}	$S_j^{\mathrm{x}} \to S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to -S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$
Ξ_{y}	$S_j^{\mathrm{x}} \to -S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to -S_j^{\mathrm{z}}$
Ξ_{z}	$S_j^{\mathrm{x}} \to -S_j^{\mathrm{x}}$	$S_j^{\mathrm{y}} \to -S_j^{\mathrm{y}}$	$S_j^{\mathrm{z}} \to S_j^{\mathrm{z}}$

Heisenberg Hamiltonian $H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$ is invariant

Conjecture (Haldane): the model has a unique gapped ground state only when S is an integer

Half-odd-integral S cases are proved (LSM theorem)

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

index for a unique gapped ground state

We want define an index for a G-invariant unique gapped ground state ω on infinite chain

ω $\qquad$ $g \in G$

transformation corresponding to g

invariant ...

fictitious "cut" at site j $\qquad$ ω_j

ω

g

the state restricted on the half-infinite chain may exhibit nontrivial transformation property → index Ind_j

Easy example dimerized state

$\mathbb{Z}_2 \times \mathbb{Z}_2$ **invariant** $H = \sum_{j\in\mathbb{Z}} \boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j+1}$ **with** $S = \frac{1}{2}$

unique gapped g.s. $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} \frac{|\uparrow\rangle_{2j}|\downarrow\rangle_{2j+1} - |\downarrow\rangle_{2j}|\uparrow\rangle_{2j+1}}{\sqrt{2}}$

invariant

Ind_2 **should be** 0

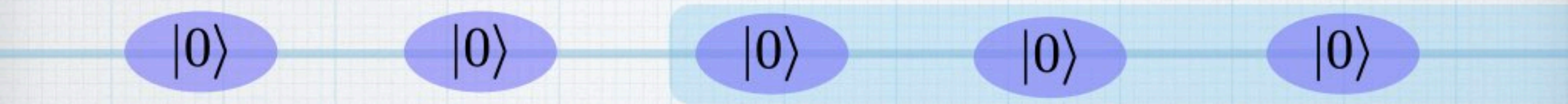
free $S = 1/2$ **spin at the edge**

Ind_1 **should be** 1

Examples in $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1$ chains

$$\mathfrak{h}_j = \mathrm{span}(|0\rangle_j, |+\rangle_j, |-\rangle_j)$$

$H = \sum_{j\in\mathbb{Z}} (S_j^{\mathrm{z}})^2$ **unique gapped g.s.** $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} |0\rangle_j$



Ind_j **should be** 0

AKLT model $H = \sum_{j\in\mathbb{Z}} \{ \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2 \}$

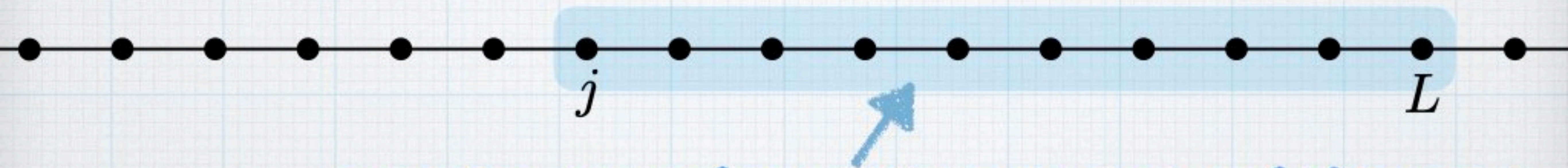
unique gapped g.s. = VBS state

effective $S = 1/2$ **edge spin**

Ind_j **should be** 1

How do we define such an index for a unique gapped g.s.?

a large but finite system

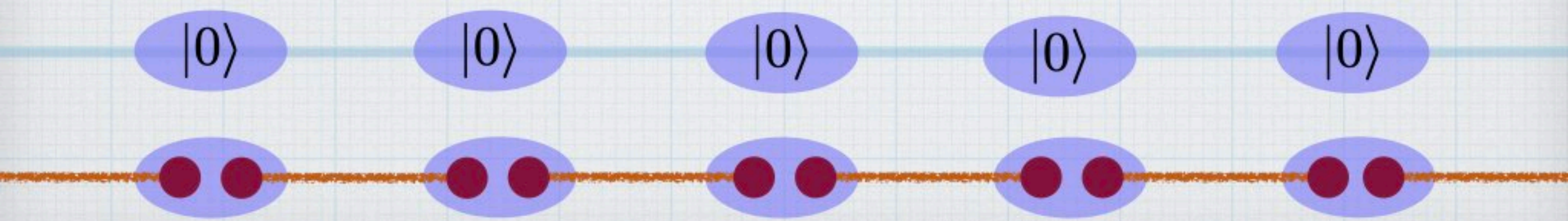


examine transformation property of this part

$\mathrm{Ind}_j = \sum_{k=j}^{L} \mathrm{ind_k}$ $\quad$ ind_k index for the single spin at site k

does not reflect the property of the g.s.!

we always have $\mathrm{Ind}_j = 0$ for $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric $S=1$ chain



Index for matrix product states

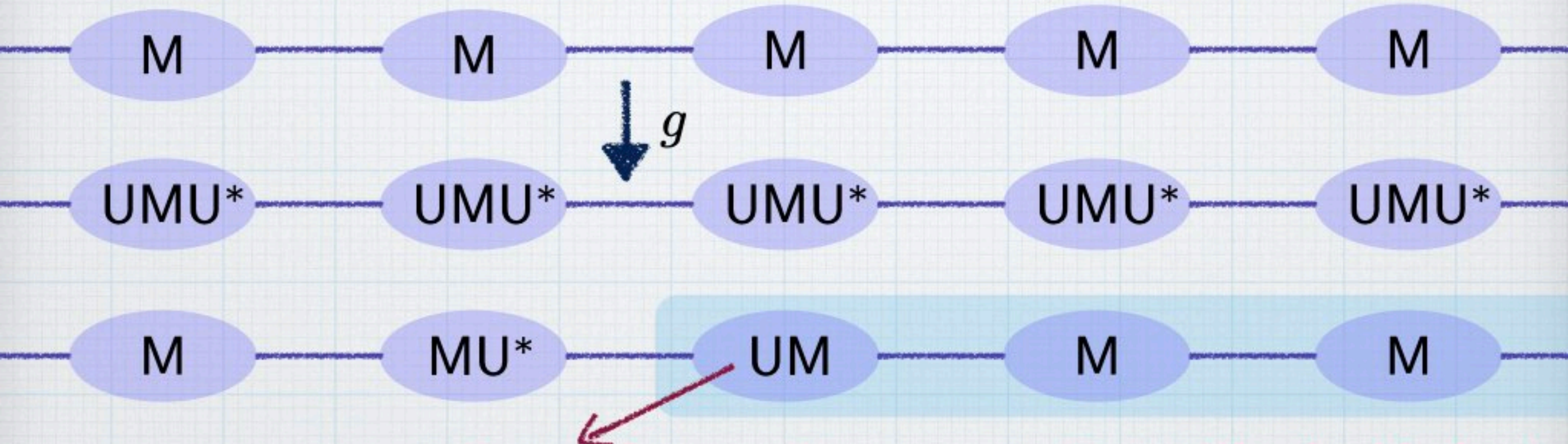
matrix product states (MPS) = finitely correlated states

Fannes, Nachtergaele, Werner 1989, 1992

G-invariant injective MPS

$$|\Phi\rangle = \sum_{\sigma_1,\dots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\dots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\dots,\sigma_L\rangle$$

matrices transform as $\mathsf{M}^{\sigma} \;\rightarrow\; \mathsf{U}_g\mathsf{M}^{\sigma}\mathsf{U}_g^*$



transformation property in the half chain

U_g proj. rep. of G ⟶ meaningful index $\mathrm{Ind} \in \mathrm{H}^2(G, \mathrm{U}(1))$

Pollmann, Turner, Berg, Oshikawa 2010

Perez-Garcia, Wolf, Sanz, Verstraete, and Cirac 2008

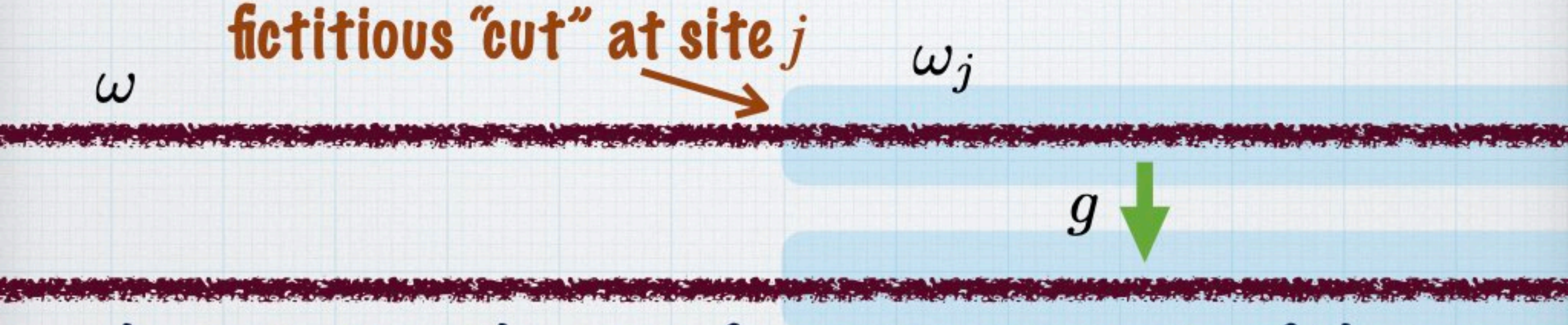
Fannes, Nachtergaele, Werner 199?

Matsui 2001

Ogata index for a general G-invariant unique gapped ground state

Ogata 2018 (Matsui 2001, 2013)

index $\mathrm{Ind}_j^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$ for any G-invariant unique gapped g.s. (more generally, pure split state)



characterizes the transformation property of the g.s. restricted on the half-infinite chain

$$\pi_j(\mathfrak{A}_j) \subset \pi_j(\mathfrak{A}_j)'' \subset B(\mathcal{H}_j)$$
$$\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$$

operator algebraic formulation
proj. rep. on the von Neumann algebra

$$\pi_j(\mathfrak{A}_j) \xrightarrow{\text{bicommutant}} \pi_j(\mathfrak{A}_j)''$$

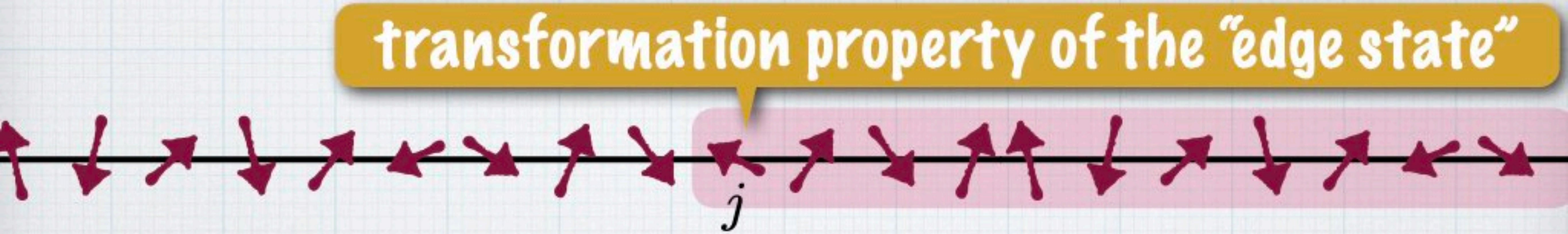
representation of the C* algebra

von Neumann algebra

basic properties of Ogata index

G-invariant unique gapped ground state ω site $j \in \mathbb{Z}$

- unique well defined index $\mathrm{Ind}_j^{\omega} \in \mathrm{H}^2(G, \mathrm{U}(1))$



- coincides with the index of Pollmann, Turner, Berg, and Oshikawa for MPS (matrix product states)

Ogata 2018

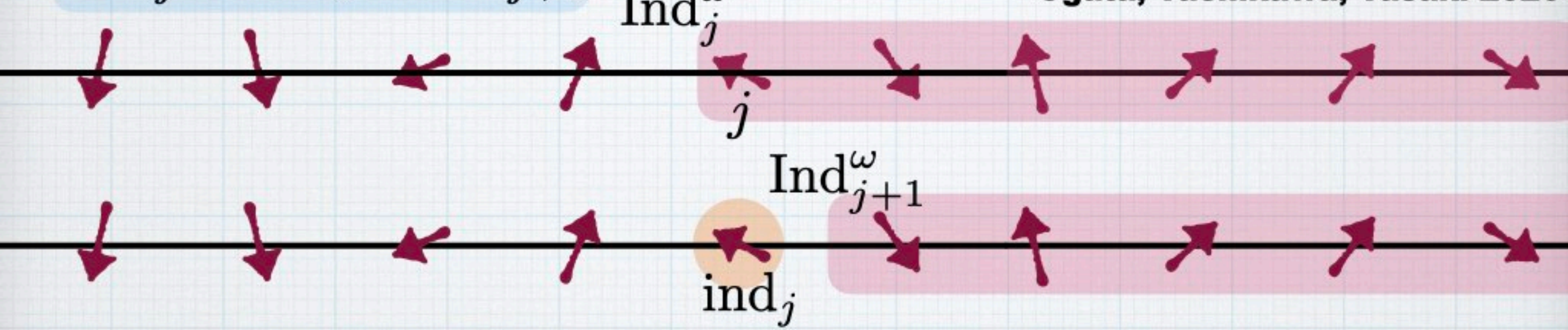
- invariant under smooth modification of G-invariant models with a unique gapped ground state

Ogata 2018

properties of Ogata index: additivity

$$\mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$$

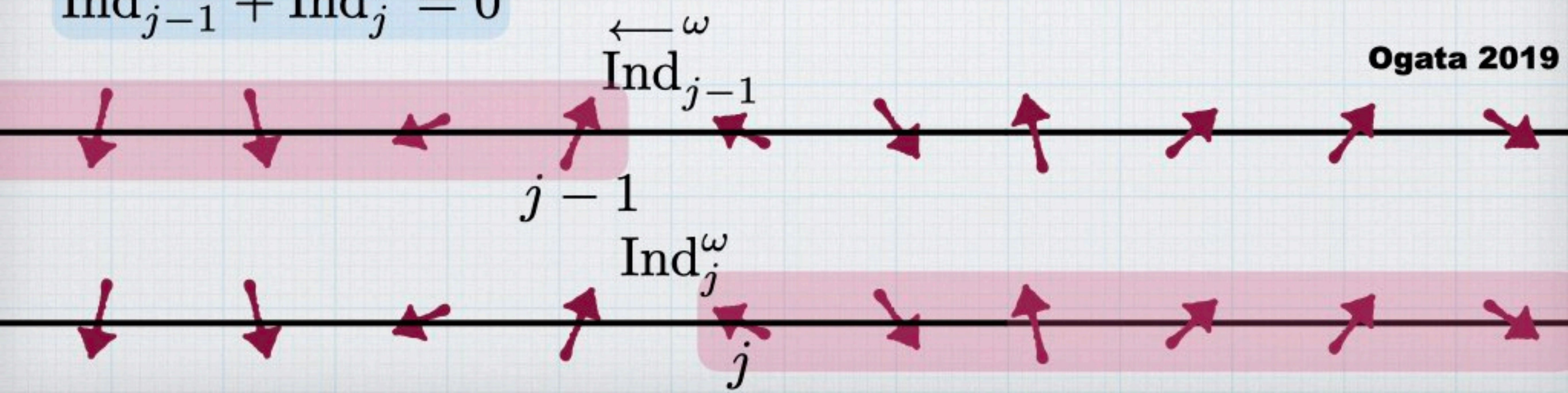
Ogata, Tachikawa, Tasaki 2020



transformation property of the ground state restricted to the left half-infinite chain

$$\overleftarrow{\mathrm{Ind}}_{j-1}^{\omega} + \mathrm{Ind}_j^\omega = 0$$

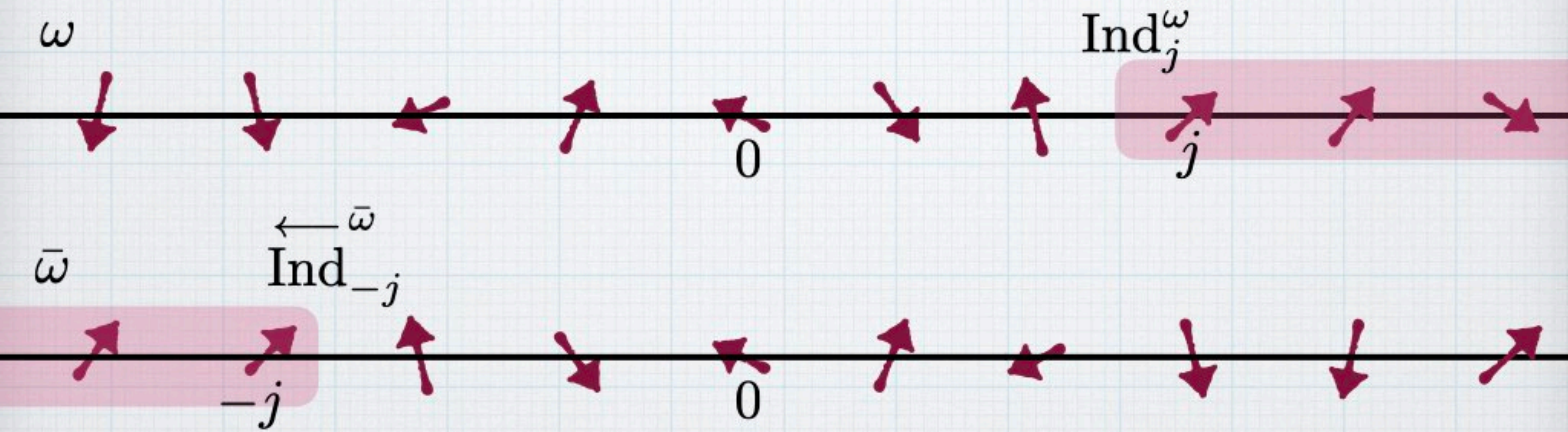
Ogata 2019



properties of Ogata index: inversion

spatial inversion $j \to -j$ $\quad \omega \to \bar{\omega}$

$$\mathrm{Ind}_j^{\bar{\omega}} = -\mathrm{Ind}_{1-j}^{\omega}$$



$$\mathrm{Ind}_j^{\omega} = \overleftarrow{\mathrm{Ind}}_{-j}^{\bar{\omega}} = -\mathrm{Ind}_{-j+1}^{\bar{\omega}}$$

trivial

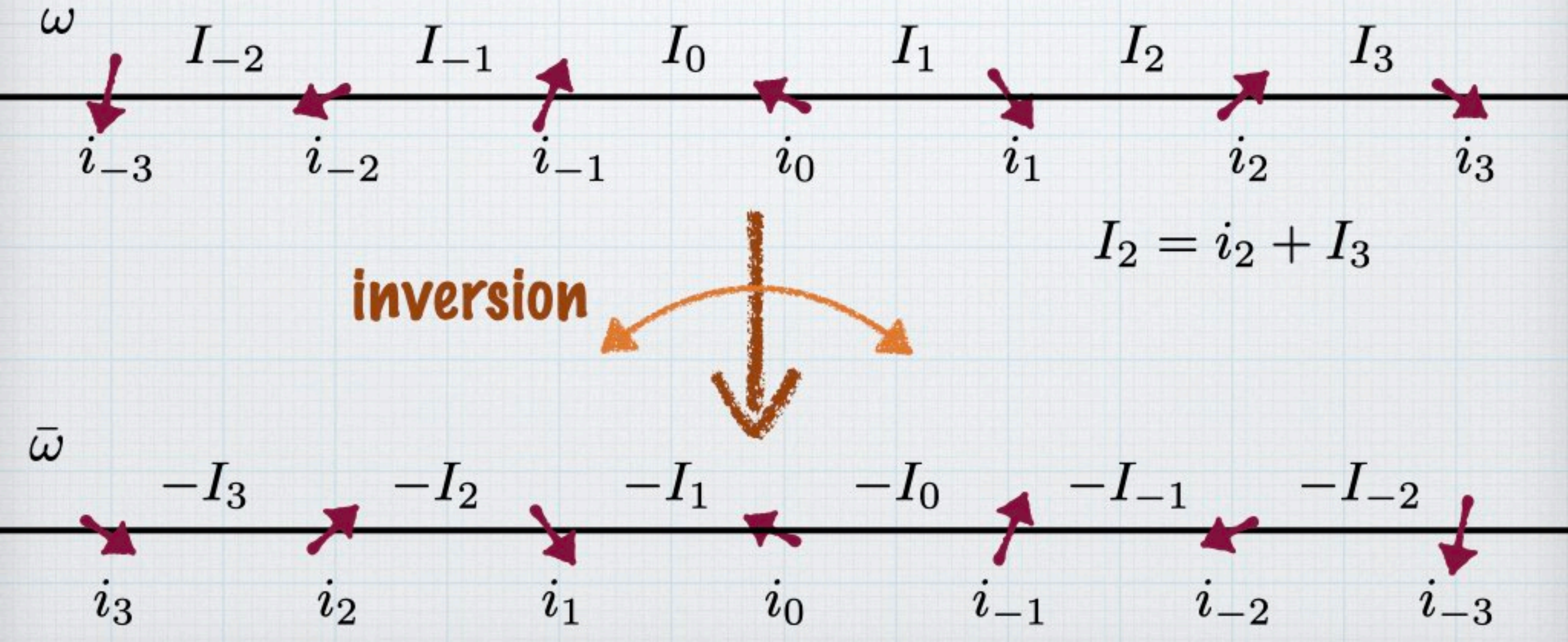
additivity

graphic notation and summary of the properties

$i_j = \mathrm{ind}_j \in \mathrm{H}^2(G, U(1))$ index for spin at j

$I_j = \mathrm{Ind}_j^{\omega} \in \mathrm{H}^2(G, U(1))$

Ogata index for ω restricted to $\{j, j+1, \ldots\}$



Index for projective representation of symmetry group

Index for a unique gapped ground state

$\boxed{\text{Application 1: LSM type theorems}}$

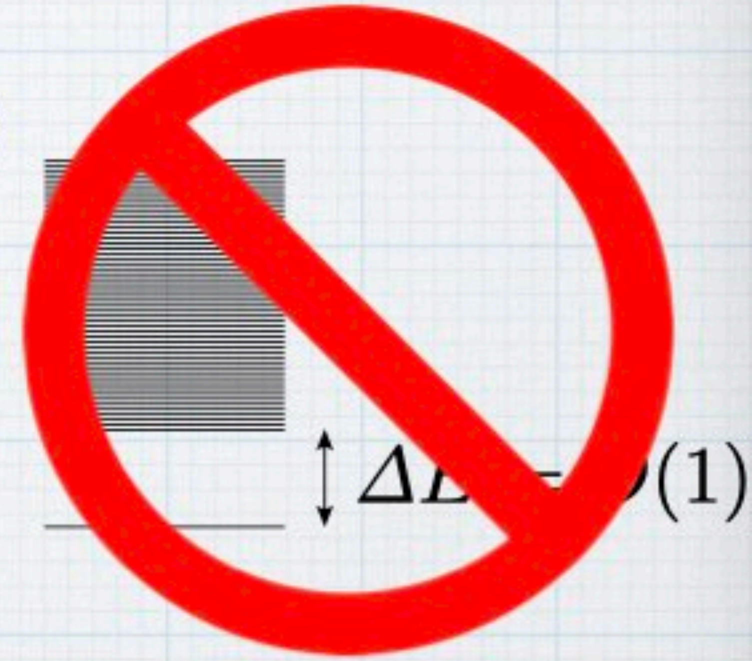
Application 2: classification of SPT

Definition of Ogata index

Lieb-Schultz-Mattis (LSM) type theorems

No-go theorems which state that certain quantum many-body systems **CANNOT** have a unique gapped ground state

symmetry ⟶ low energy properties



the original theorem and its extensions

$\mathrm{U}(1)$ invariance is essential

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

Oshikawa, Yamanaka, Affleck 1997

Oshikawa 2000, Hastings 2004, Nachtergaele, Sims 2007

Bachmann, Bols, De Roeck, Fraas 2019

recent "extensions"

similar no-go statements for models without continuous symmetry, but with some discrete symmetry

Chen, Gu, Wen 2011

Fuji 2016

Watanabe, Po, Vishwanath, Zaletel 2015

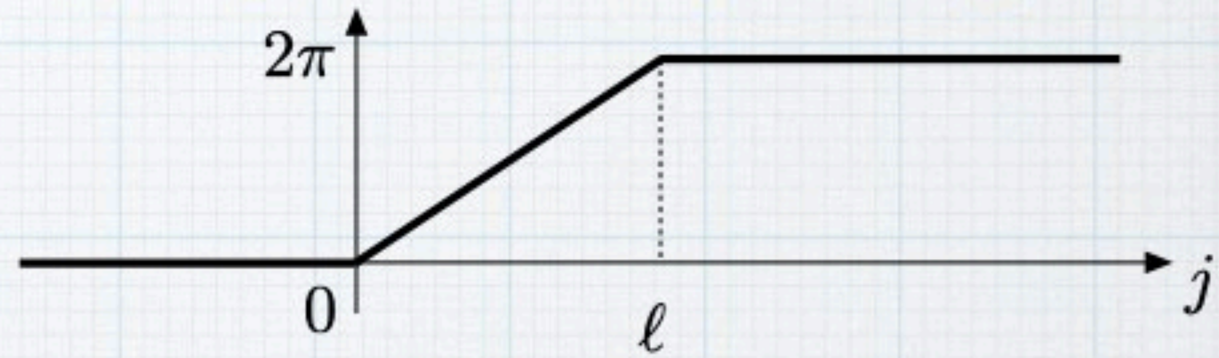
Po, Watanabe,Jian, Zaletel 2017

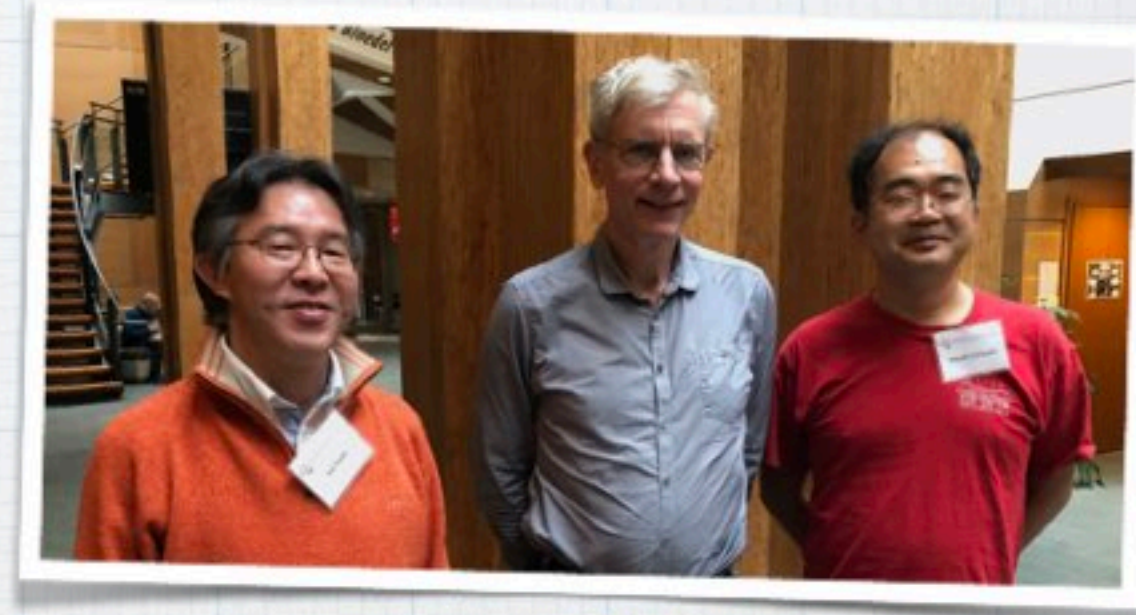
Ogata, Tasaki 2019, Ogata, Tachikawa, Tasaki 2020

Strategies of the proofs

original strategy
Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

explicit construction of low-lying states by using gradual "$\mathrm{U}(1)$ twist"





new strategy

$\exists$ a unique gapped ground state $\Longrightarrow$ a condition

$\overline{\text{a condition}}$ $\Longrightarrow$ no unique gapped ground state

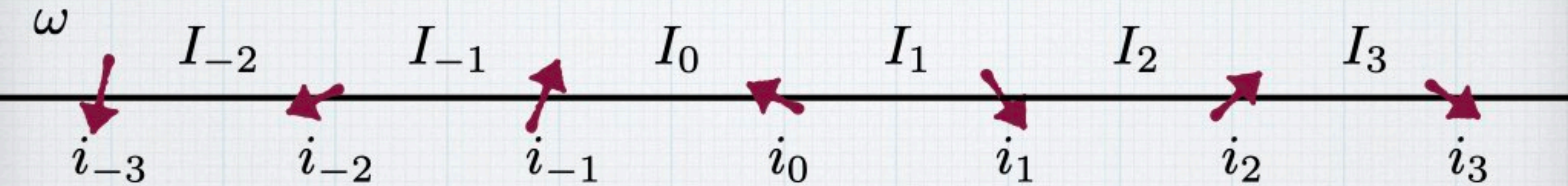
Oshikawa 2000

make full use of Ogata index

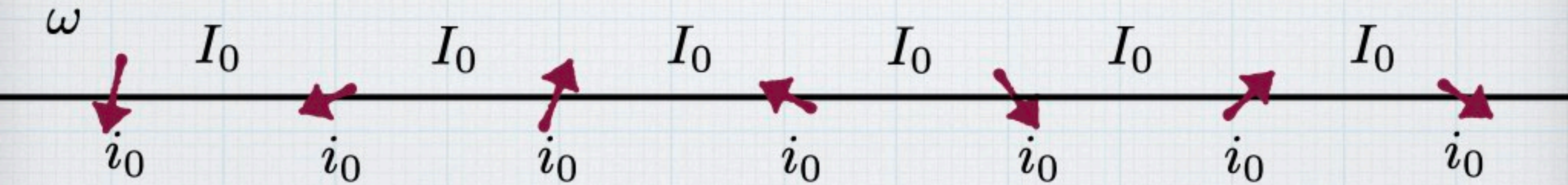
models with translational symmetry

quantum spin chain with a translation invariant and G-invariant short ranged Hamiltonian

assume that the ground state ω is unique and accompanied by a nonzero gap



$$I_2 = i_2 + I_3$$



$$I_0 = i_0 + I_0$$

$$\Downarrow$$

$$i_0 = 0$$

models with translational symmetry

quantum spin chain with a translation invariant and G-invariant short ranged Hamiltonian

assume that the ground state ω is unique and accompanied by a nonzero gap

THEOREM: If a quantum spin chain with translation invariant and G-invariant short ranged Hamiltonian has a unique gapped g.s., it must be that $\mathrm{ind}_j = 0$ for any $j \in \mathbb{Z}$

LSM type no-go theorem

COROLLARY: A quantum spin chain with translation invariant and G-invariant short ranged Hamiltonian with $\mathrm{ind}_j \neq 0$ can never have a unique gapped g.s.

Chen, Gu, Wen 2011
Watanabe, Po, Vishwanath, Zaletel 2015

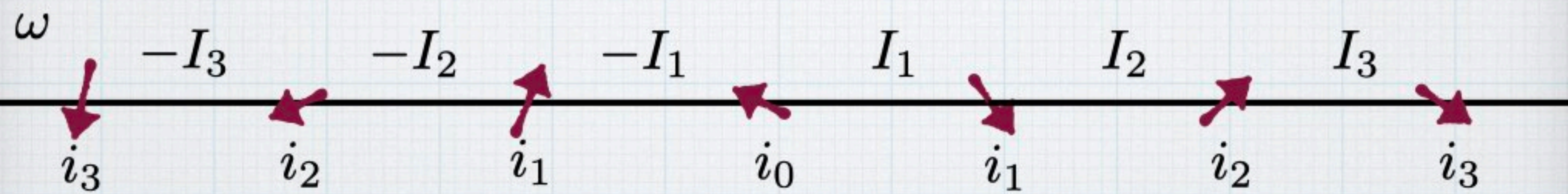
when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\qquad \mathrm{ind}_j \neq 0 \longleftrightarrow S_j = \frac{1}{2}, \frac{3}{2}, \ldots$

models with inversion symmetry

quantum spin chain with a G-invariant short ranged Hamiltonian that is invariant under inversion about the origin

translation invariance is not assumed

assume that the ground state ω is unique and accompanied by a nonzero gap



$$-I_1 = i_0 + I_1 \qquad i_0 = -2I_1$$

THEOREM: If a quantum spin chain with G-invariant and reflection invariant short ranged Hamiltonian has a unique gapped g.s., it must be that $\mathrm{ind}_0 \in 2\,\mathrm{H}^2(G, \mathrm{U}(1))$

models with inversion symmetry

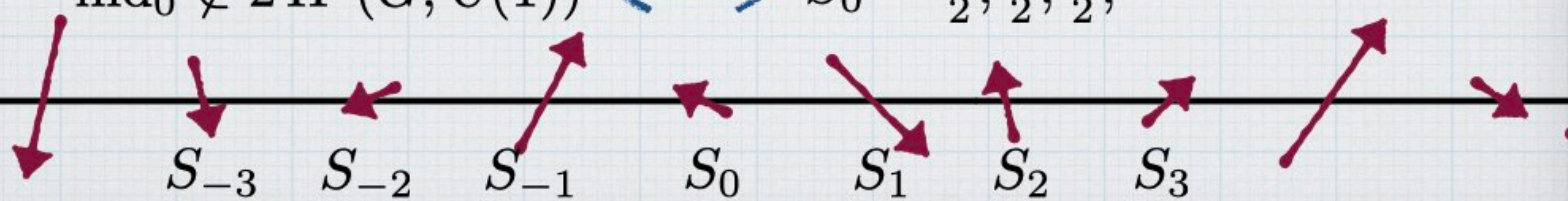
THEOREM: If a quantum spin chain with G-invariant and reflection invariant short ranged Hamiltonian has a unique gapped g.s., it must be that $\mathrm{ind}_0 \in 2\,\mathrm{H}^2(G, \mathrm{U}(1))$

LSM type no-go theorem

COROLLARY: A quantum spin chain with $\mathrm{ind}_0 \notin 2\,\mathrm{H}^2(G, \mathrm{U}(1))$ and G-invariant short ranged Hamiltonian which is invariant under reflection around the origin can never have a unique gapped ground state

when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\mathrm{ind}_0 \notin 2\,\mathrm{H}^2(G, \mathrm{U}(1)) \Longleftrightarrow S_0 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$$

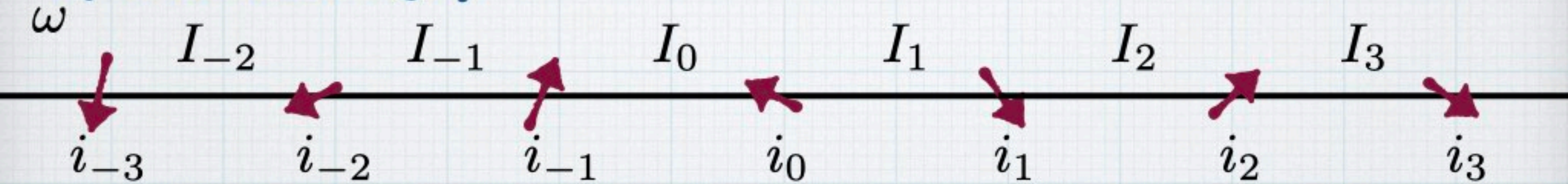


Fuji 2016, Po, Watanabe, Jian, Zaletel 2017

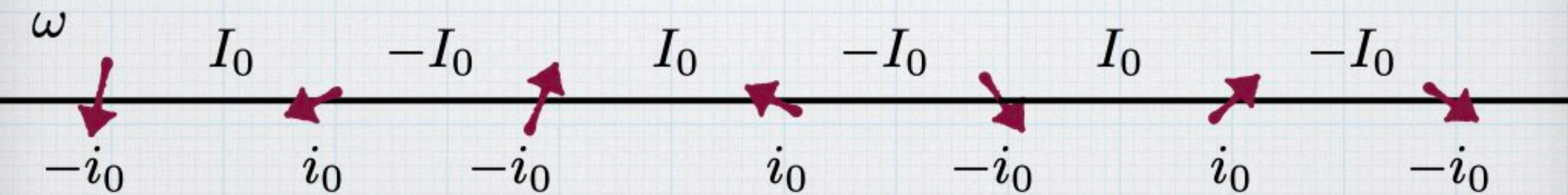
more complicated symmetry

quantum spin chain with a G-invariant short ranged Hamiltonian that is invariant under { translation + a fixed automorphism a of G }

assume that the ground state ω is unique and accompanied by a nonzero gap



$$I_2 = i_2 + I_3$$



$$I_0 = i_0 - I_0$$

$$\Downarrow$$

$$i_0 = 2I_0$$

a simple example in which a flips the sign of the indices

Index for projective representation of symmetry group

Index for a unique gapped ground state

Application 1: LSM type theorems

$\boxed{\text{Application 2: classification of SPT}}$

Definition of Ogata index

Classification of unique gapped g.s.

H_0, H_1 G-invariant Hamiltonians of the same spin chain
ω_0, ω_1 unique gapped ground states

definition

ω_0 and ω_1 are smoothly connected if and only if there exists a family of Hamiltonians H_s with $0 \leq s \leq 1$ such that

H_s has a unique gapped g.s. ω_s for each s

ω_s depends smoothly on s

Conjecture: Any unique gapped ground states of the same spin chain are smoothly connected

Chen, Gu, Wen 2011

Theorem: This is true for matrix product states

Ogata 2017

There is only one phase!

Classification of unique gapped g.s.

H_0, H_1 G-invariant Hamiltonians of the same spin chain
ω_0, ω_1 unique gapped ground states

definition

ω_0 and ω_1 are smoothly connected via G-symmetric models if and only if there exists a family of G-invariant Hamiltonians H_s with $0 \le s \le 1$ such that

H_s has a unique gapped g.s. ω_s for each s

ω_s depends smoothly on s

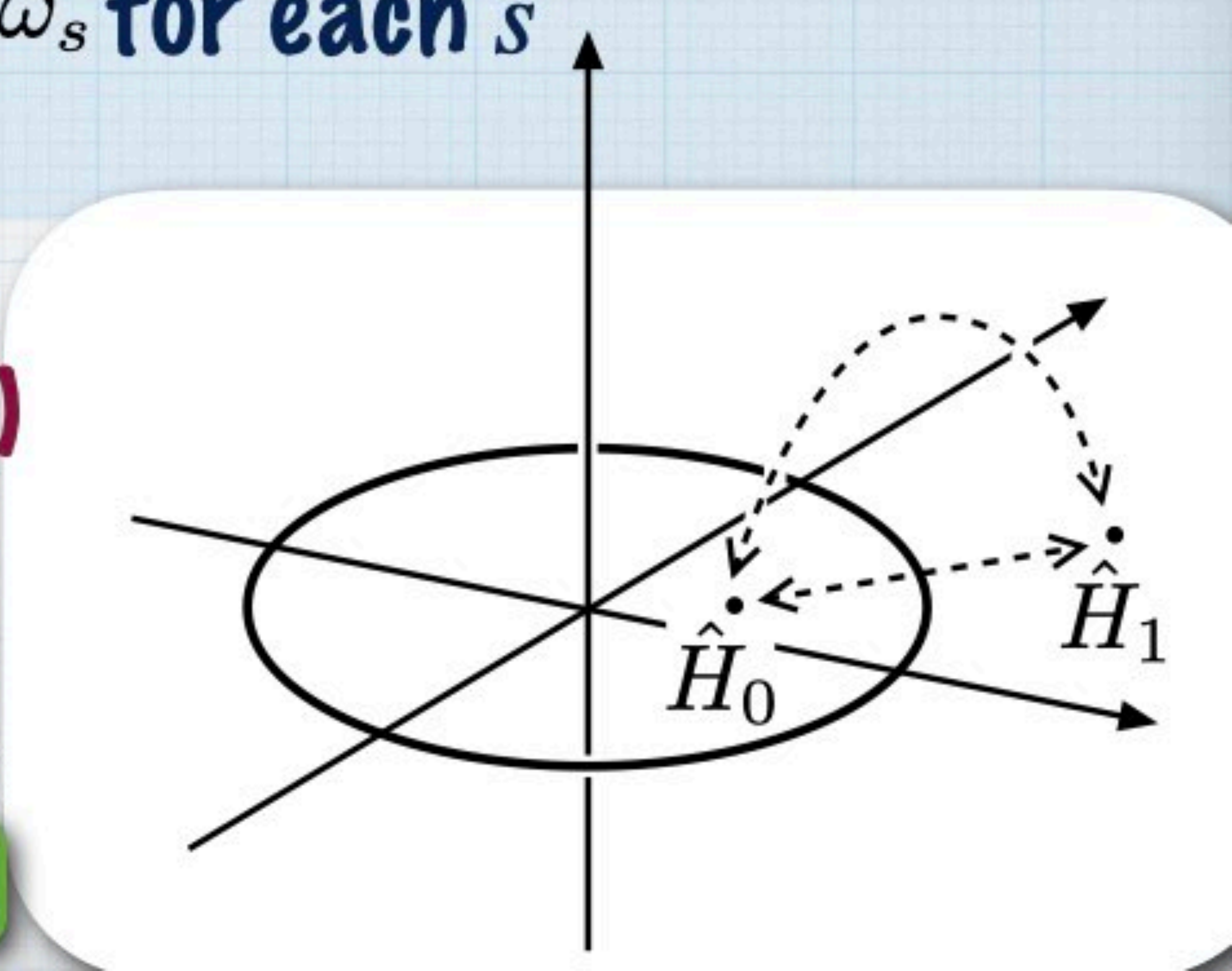
the classification of the equivalence classes (SPT phases) is given by the index

$$\mathrm{Ind} \in \mathrm{H}^2(G, \mathrm{U}(1))$$

MPS

Pollmann, Turner, Berg, Oshikawa 2010
Gu, Wen 2009
Chen, Gu, Wen 2011

RG fixed points



Classification of unique gapped g.s.

H_0, H_1 G-invariant Hamiltonians of the same spin chain
ω_0, ω_1 unique gapped ground states

definition

ω_0 and ω_1 are smoothly connected via G-symmetric models if and only if there exists a family of G-invariant Hamiltonians H_s with $0 \le s \le 1$ such that

H_s has a unique gapped g.s. ω_s for each s

ω_s depends smoothly on s

$\mathrm{Ind}_j^{\omega} \in \mathrm{H}^2(G, \mathrm{U}(1))$ index for a unique gapped g.s. ω

essential theorem Ogata 2018

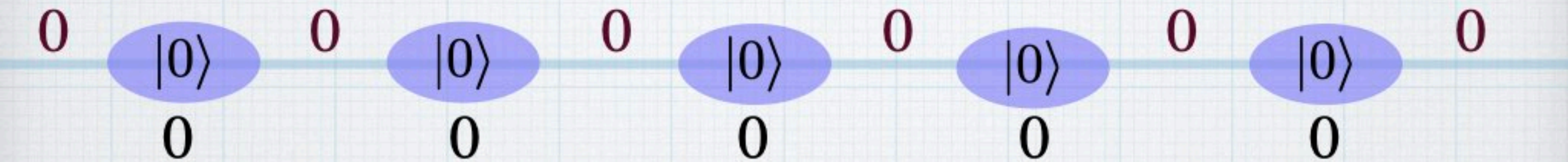
Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then ω_0 and ω_1 can never be smoothly connected via G-symmetric models

Ind_0^{ω} determines Ind_j^{ω} for all j because $\mathrm{Ind}_j^{\omega} = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^{\omega}$

example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1$ chains

Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then ω_0 and ω_1 can never be smoothly connected via G-symmetric models

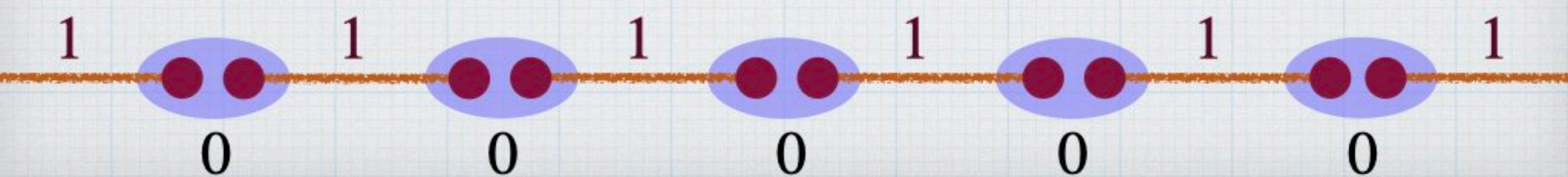
$H = \sum_{j\in\mathbb{Z}} (S_j^{\mathrm{z}})^2$ unique gapped g.s. $\Phi_{\mathrm{GS}} = \bigotimes_{j\in\mathbb{Z}} |0\rangle_j$



these ground states are never connected smoothly via $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric models

AKLT model $H = \sum_{j\in\mathbb{Z}} \{\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + \frac{1}{3}(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1})^2\}$

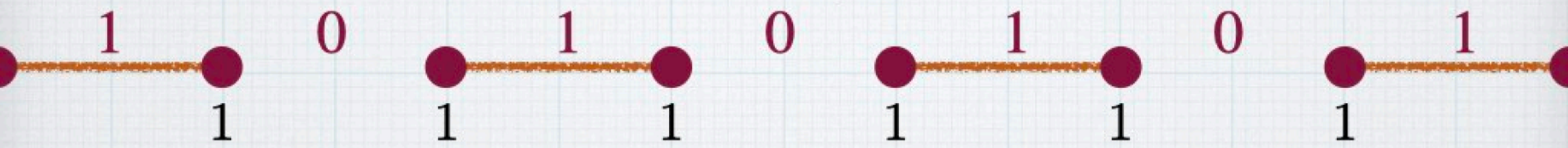
unique gapped g.s. = VBS state



example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant $S = 1/2$ chains

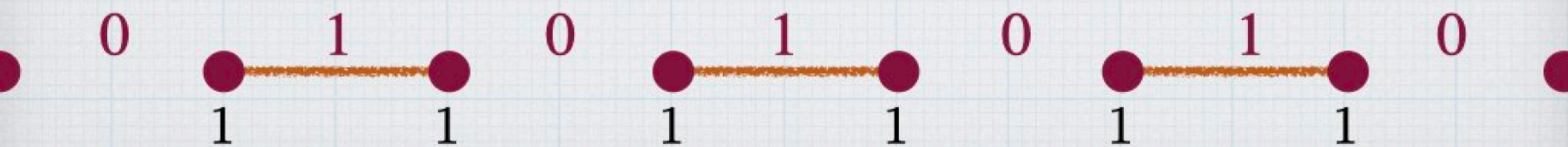
Theorem: If $\mathrm{Ind}_0^{\omega_0} \neq \mathrm{Ind}_0^{\omega_1}$, then ω_0 and ω_1 can never be smoothly connected via G-symmetric models

$H_0 = \sum_{j \in \mathbb{Z}} \boldsymbol{S}_{2j} \cdot \boldsymbol{S}_{2j+1}$ $\quad$ $\Phi_0 = \bigotimes_{j \in \mathbb{Z}} \dfrac{|\uparrow\rangle_{2j} |\downarrow\rangle_{2j+1} - |\downarrow\rangle_{2j} |\uparrow\rangle_{2j+1}}{\sqrt{2}}$



these ground states are never connected smoothly via $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric models

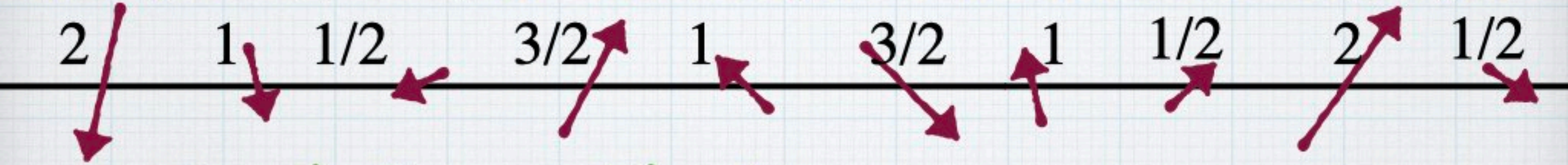
$H_1 = \sum_{j \in \mathbb{Z}} \boldsymbol{S}_{2j-1} \cdot \boldsymbol{S}_{2j}$ $\quad$ $\Phi_1 = \bigotimes_{j \in \mathbb{Z}} \dfrac{|\uparrow\rangle_{2j-1} |\downarrow\rangle_{2j} - |\downarrow\rangle_{2j-1} |\uparrow\rangle_{2j}}{\sqrt{2}}$



general classification by Ogata

for any G-invariant quantum spin chains (without translation invariance) unique gapped ground states are classified by a single index $\mathrm{Ind}_0^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$

example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant spin chain with site-dep. spins



Ind_j^ω at each site can either be

0 0 0 1 0 0 1 1 0 0

0 0 1 1 0 1 0 1 0 1

or

1 1 1 0 1 1 0 0 1 1

0 0 1 1 0 1 0 1 0 1

Ind_0^ω determines Ind_j^ω for all j because $\mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$

general classification by Ogata

for any G-invariant quantum spin chains (without translation invariance) unique gapped ground states are classified by a single index $\mathrm{Ind}_0^\omega \in \mathrm{H}^2(G, \mathrm{U}(1))$

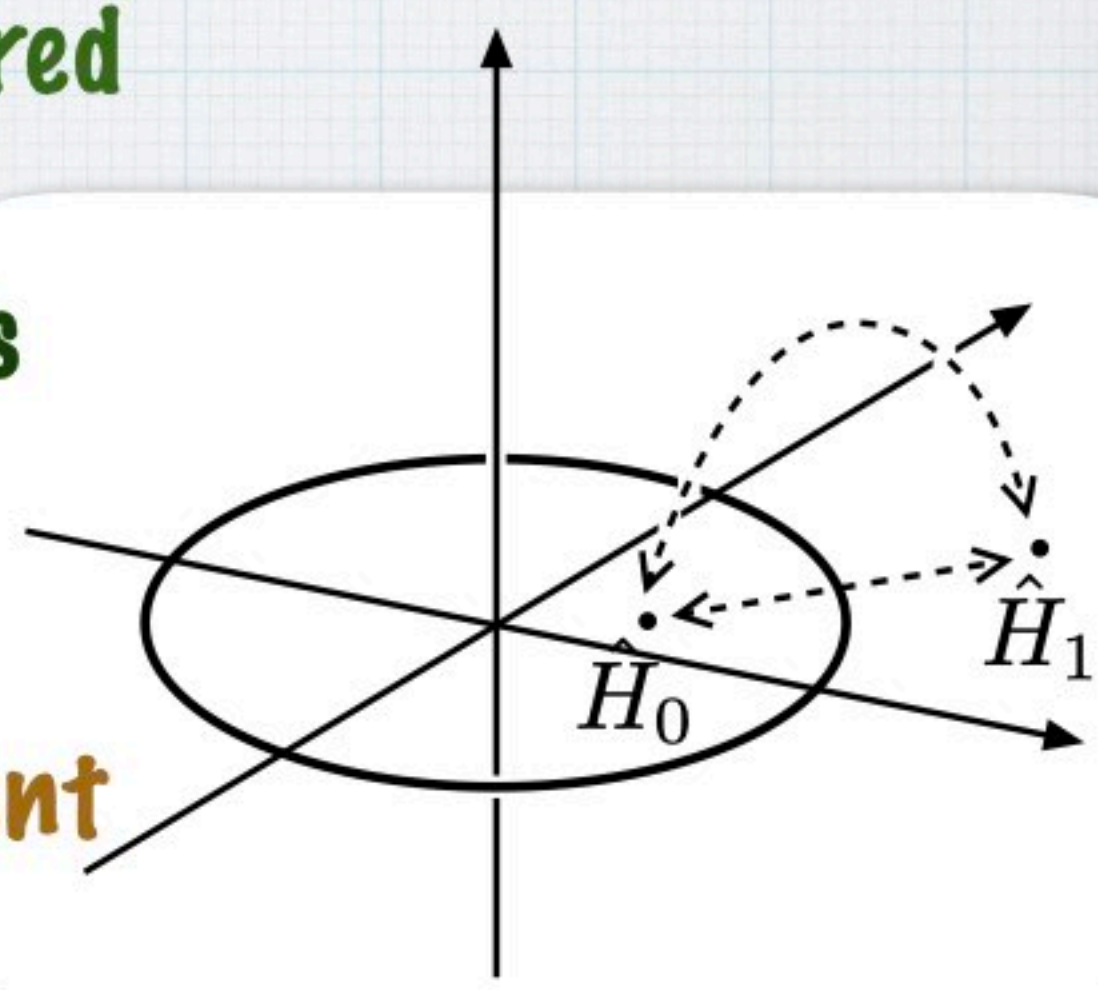
a strict extension of the known classifications

Pollmann, Turner, Berg, Oshikawa 2010

works for injective MPS (matrix product states)
translation invariance is required

Chen, Gu, Wen 2011

classification of RG fixed points
some uniformity is assumed



Remark: it is possible that different "phases" have the same index

the concept of full SPT (symmetry protected topological) order

definition

G-invariant unique gapped ground state ω has full SPT order if and only if $\mathrm{Ind}_j^\omega \neq 0$ for any $j \in \mathbb{Z}$

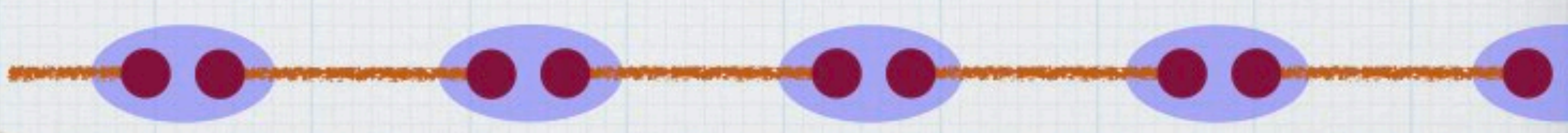
S_j^ω entanglement entropy between $\{\ldots, j-1\}$ and $\{j, \ldots\}$

Ogata 2018

Theorem: If ω has full SPTO then $S_j^\omega \geq \log 2$ for any $j \in \mathbb{Z}$

entanglement enforced by symmetry!

Pollmann, Turner, Berg, Oshikawa 2010



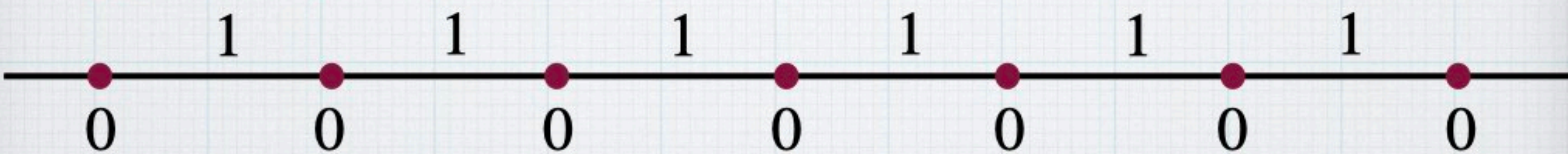
https://www.irasutoya.com/2014/02/blog-post_2233.html

a necessary condition for full SPT order

definition

G-invariant unique gapped ground state ω has full SPT order if and only if $\mathrm{Ind}_j^\omega \neq 0$ for any $j \in \mathbb{Z}$

Theorem: If $\mathrm{H}^2(G, \mathrm{U}(1)) = \{0, 1\}$ then full SPTO is possible only for a spin chain with $\mathrm{ind}_j = 0$ for all $j \in \mathbb{Z}$



$$\mathrm{Ind}_j^\omega = \mathrm{ind}_j + \mathrm{Ind}_{j+1}^\omega$$

We find Haldane phase (protected by on-site symmetry) only for integer spins!

Remark: In a translationally invariant model full SPTO is possible only when $\mathrm{ind}_j = 0$ for all $j \in \mathbb{Z}$ (LSM theorem)

enforcement of full SPT order

SPT-LSM theorem

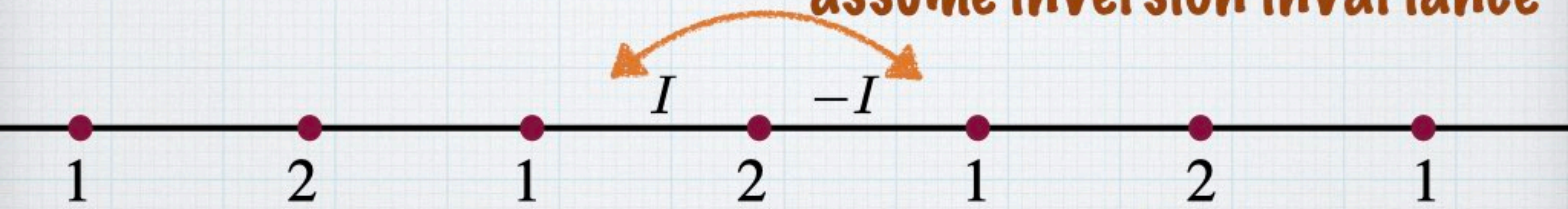
Jiang, Cheng, Qi, Lu 2019

Haruki Watanabe, private communication

simplest example: unique gaped ground state invariant under $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathrm{SU}(3)$

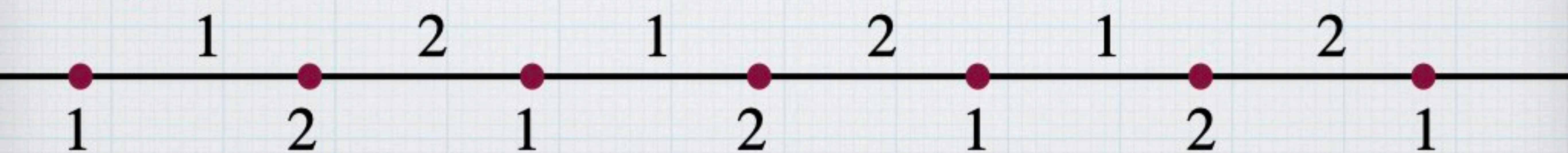
$\mathrm{H}^2(G, \mathrm{U}(1)) = \mathbb{Z}_3 = \{0, 1, 2\}$

assume inversion invariance

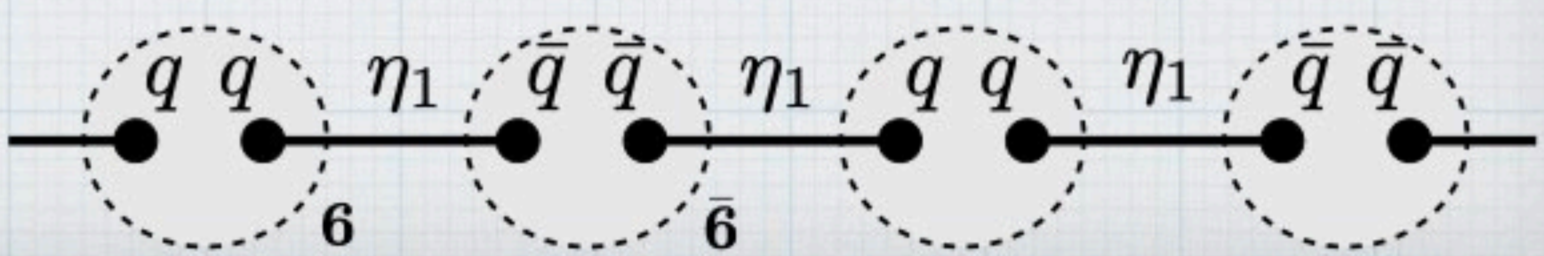


$I = 2 - I \longrightarrow I = 1$

the ground state has full SPT order



SU(3) **AKLT state**



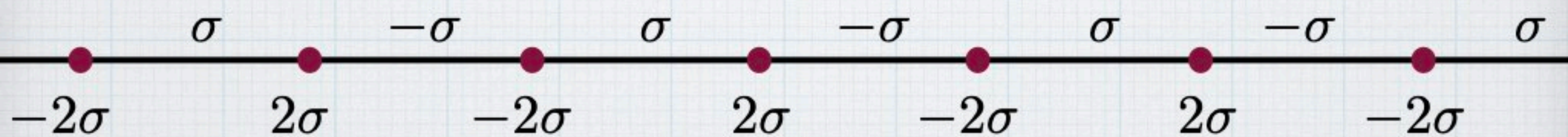
$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$ $\quad$ $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{6}} \oplus \mathbf{3}$

enforcement of full SPT order

Jiang, Cheng, Qi, Lu 2019

Haruki Watanabe, private communication

Theorem: Let $\sigma \in \mathrm{H}^2(G, \mathrm{U}(1))$ be such that $2\sigma \neq 0$, and consider a "spin chain" with $\mathrm{ind}_j = 2\sigma$ for even j and $\mathrm{ind}_j = -2\sigma$ for odd j, and assume that the model is invariant under inversion about the origin. Then the ground state of the model has full SPT order if it is unique and gapped.



inversion symmetry can be replaced by symmetry under {translation + $\mathrm{ind} \to -\mathrm{ind}$}

Index for projective representation of symmetry group

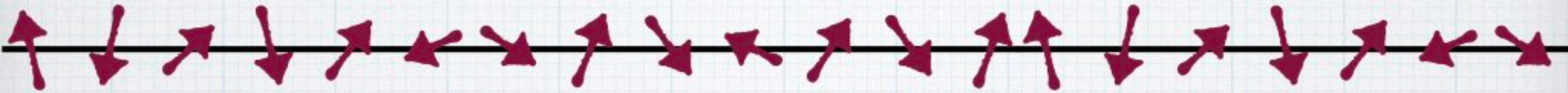
Index for a unique gapped ground state

Application 1: LSM type theorems

Application 2: classification of SPT

Definition of Ogata index

General quantum spin chain



$\mathfrak{h}_j$ **Hilbert space at site** $j \in \mathbb{Z}$ $\qquad \dim(\mathfrak{h}_j) \le d_0$

C^*-**algebra** $\quad \mathfrak{A} = \overline{\{\text{all local operators}\}}$

G **symmetry group (finite group)**

$u_g^{(j)}$ **unitary on** $\mathfrak{h}_j$

projective representation with index $\mathrm{ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

$*$-**automorphism on** $\mathfrak{A}$

$$\Xi_g(A) = \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right) A \left(\bigotimes_{j=-L}^{L} u_g^{(j)}\right)^*$$

for $g \in G$ **and a local operator** A

$$\Xi_g \circ \Xi_h = \Xi_{gh}$$

G-invariant Hamiltonian and a unique gapped g.s.

formal expression

G-invariant short ranged Hamiltonian $H = \sum_{j\in\mathbb{Z}} h_j$

$h_j = h_j^*$ acts only on $\bigotimes_{k;|k-j|\le r_0} \mathfrak{h}_k$

$\Xi_g(h_j) = h_j$ for any $j \in \mathbb{Z}$ and $g \in G$

basic assumption: the ground state ω of H is unique and accompanied by a nonzero energy gap

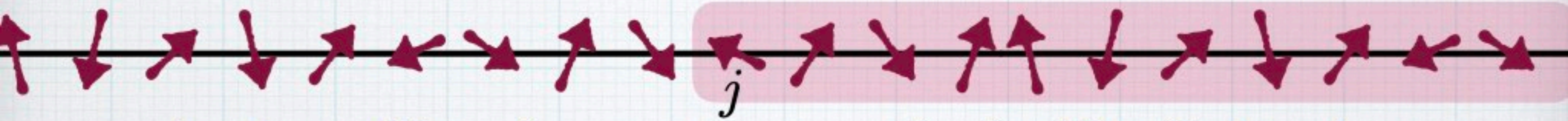
$$\omega(A) = \lim_{L\uparrow\infty} \langle \Phi_{\mathrm{GS}}^{(L)}, A\Phi_{\mathrm{GS}}^{(L)} \rangle$$

Def: a state is a linear function $\omega : \mathfrak{A} \to \mathbb{C}$ such that $\omega(I) = 1$ and $\omega(A^*A) \ge 0$ for any $A \in \mathfrak{A}$

Def: ω is a g.s. if $\omega(A^*[H, A]) \ge 0$ for any local operator A

Def: a unique g.s. ω is accompanied by a nonzero gap if there is $\gamma > 0$ and $\omega(A^*[H, A]) \ge \gamma\, \omega(A^*A)$ for any A s.t. $\omega(A) = 0$

GNS Hilbert space for half infinite chain



C^*-algebra of local operators on the half-infinite chain

$$\mathfrak{A}_j = \overline{\{\text{all local operators on } \{j, j+1, \ldots\}\}} \qquad j \in \mathbb{Z}$$

ω_j restriction of the ground state ω on $\mathfrak{A}_j$

$\mathfrak{A}_j$ and ω_j $\xrightarrow{\text{GNS construction}}$ $(\mathcal{H}_j, \pi_j, \Omega_j)$ $\Omega_j \in \mathcal{H}_j$

representation $\pi_j : \mathfrak{A}_j \to B(\mathcal{H}_j)$

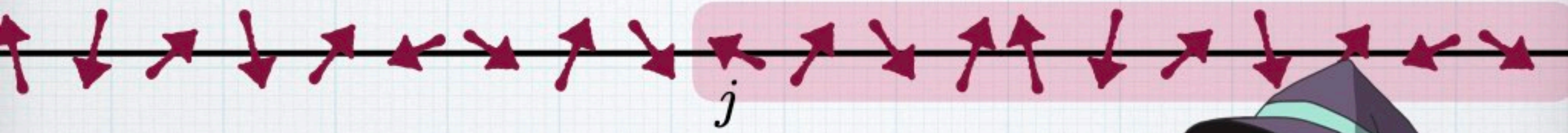
$\omega_j(A) = \langle \Omega_j, \pi_j(A)\, \Omega_j \rangle$ $\quad \{\pi_j(A)\, \Omega_j \,|\, A \in \mathfrak{A}_j\}$ is dense in $\mathcal{H}_j$

noting the G-invariance $\omega_j(\Xi_g(A)) = \omega_j(A)$, we can define unitary U_g on $\mathcal{H}_j$ by $U_g\, \pi_j(A)\, \Omega_j = \pi_j(\Xi_g(A))\, \Omega_j$ for $A \in \mathfrak{A}_j$

but ... $U_g U_h = U_{gh}$ $\qquad$ genuine rep. = trivial proj. rep.

this is not yet what we want!

von Neumann algebra for half infinite chain



$\pi_j(\mathfrak{A}_j) \xrightarrow{\text{bicommutant}} \pi_j(\mathfrak{A}_j)''$

representation of the C* algebra

von Neumann algebra

$\pi_j(\mathfrak{A}_j) \subset \pi_j(\mathfrak{A}_j)'' \subset B(\mathcal{H}_j)$ ← the set of all bounded operators on $\mathcal{H}_j$

when ω is a unique gapped ground state $\pi_j(\mathfrak{A}_j)''$ is a type-I factor, which is the most well-behaved von Neumann algebra Matsui 2013

then $\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$ for some Hilbert space $\tilde{\mathcal{H}}_j$

illustration by Chisato Naruse

proj. rep. on half infinite chain Matsui 2001

$\pi_j(\mathfrak{A}_j)'' \cong B(\tilde{\mathcal{H}}_j)$ for some Hilbert space $\tilde{\mathcal{H}}_j$

one can construct a projective rep. $\tilde{U}_g$ of G on $\tilde{\mathcal{H}}_j$

the corresponding index $\mathrm{Ind}_j \in \mathrm{H}^2(G, \mathrm{U}(1))$

rough idea of the construction

$\pi_j(\mathfrak{A}_j)$ is invariant under the action of $U_g(\cdot)U_g^*$

define $*$-automorphism Γ_g on $B(\tilde{\mathcal{H}}_j)$ by

$$\Gamma_g(X) = \varphi(U_g\,\varphi^{-1}(X)\,U_g^*) \qquad \pi_j(\mathfrak{A}_j)'' \xrightarrow{\varphi} B(\tilde{\mathcal{H}}_j)$$

it holds that $\Gamma_g\Gamma_h = \Gamma_{gh}$

Wigner's theorem guarantees that there is a unitary $\tilde{U}_g$ on $\tilde{\mathcal{H}}_j$ such that $\Gamma_g(X) = \tilde{U}_g\,X\,\tilde{U}_g^*$

Summary

- Ogata index is defined for any G-invariant unique gapped g.s., and characterizes the transformation property of the g.s. restricted on the half-infinite chain
- with Ogata index, we have unified simple proofs of the generalized Lieb-Schultz-Mattis theorems for quantum spin chains with only discrete symmetry
- Ogata index provides us with a classification of G-invariant unique gapped g.s. in quantum spin chains that strictly extends the classifications of Pollmann-Turner-Berg-Oshikawa and Chen-Gu-Wen
- the new classification suggest the natural notion of full symmetry protected topological order