Rigorous Index Theory for One-Dimensional Interacting Topological Insulators

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brief introduction @ YouTube / November 2021

non-interacting topological insulators

almost complete classification in terms of the "periodic table"

Ryu, Schnyder, Furusaki, Ludwig 2010, Kitaev 2009

mathematically rigorous index theories

interacting topological insulators

complete classification is not yet known

Kitaev 2001, Hatsugai 2006, Guo, Shen 2011, Fidkowski Kitaev 2011,
Manmana, Essin, Noack, Gurarie 2012, Wang, Xu, Wang, Wu 2015,
Kapustin, Thorngren, Turzillo Wang 2015,
Shiozaki, Shapourian, Ryu 2017, Matsugatani, Ishiguro, Shiozaki, Watanabe 2018,
Ono, Trifunovic, Watanabe 2019, Kang, Shiozaki, Cho 2019,
Wheeler, Wagner, Hughes 2019, Lu, Ran, Oshikawa 2020
Nakamura, Masuda, Nishimoto 2021, Stehouwer 202, and many more

mathematically rigorous index theories are limited

Avron, Seiler 1985, Bachmann, Bols, De Roeck, Fraas 2019, 2021
Bourne, Schulz-Baldes 2020, Matsui 2020, Bourne, Ogata 2021, Ogata 2021

non-interacting topolo...

new rigorous index theory for a class of 1D topological insulators including the SSH model (class D)

... periodic

... Schnyder, Furusaki, Ludwig 2010, Kitaev 2009

mathematically ...

establishes the existence of a (symmetry protected) topological phase transition in the infinite system

in... topological insulators

com...

establishes the existence of a gapless edge mode when the topological index is nonzero

... Noack, Gurarie 2012, Wang, Xu, Wang, Wu 2015,
Kapustin, Thorngren, Turzillo, Wang 2015
Shiozak... be 2018,

proof is very elementary and simple!!

Ono, Tri...
Wheeler, Wagner, Hughes 2019, Lu, Ran, Oshikawa 2020
Nakamura, Masuda, Nishimoto 2021, Stehouwer 202, and many more

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topological index in the SSH model

Su-Schrieffer-Heeger (SSH) model

Su, Schrieffer, Heeger 1979

non-interacting model at half-filling with Hamiltonian

$$\hat{H}_s^{\mathrm{SSH}} = \sum_{j\in\mathbb{Z}} \{(1-s)(\hat{c}^\dagger_{2j}\hat{c}_{2j+1} + \mathrm{h.c.}) + s(\hat{c}^\dagger_{2j-1}\hat{c}_{2j} + \mathrm{h.c.})\}$$

$s \in [0,1]$ **model parameter**

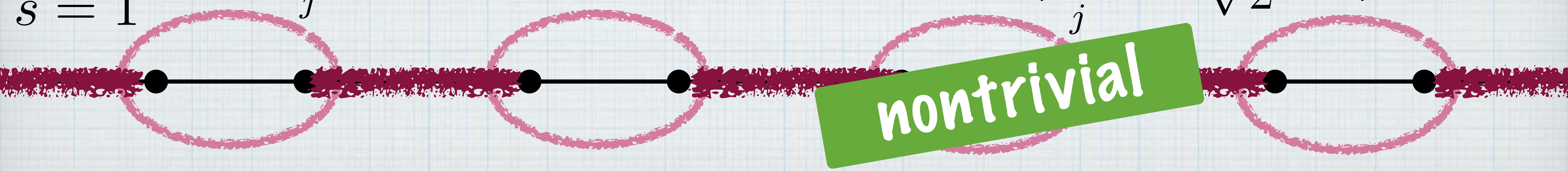
$\{2j, 2j+1\}$ **forms a unit cell**

$$\hat{H}_0^{\mathrm{SSH}} = \sum_j (\hat{c}^\dagger_{2j}\hat{c}_{2j+1} + \mathrm{h.c.}) \qquad |\Phi_{\mathrm{GS},0}\rangle = \Big(\prod_j \frac{\hat{c}^\dagger_{2j} - \hat{c}^\dagger_{2j+1}}{\sqrt{2}}\Big)|\Phi_{\mathrm{vac}}\rangle$$

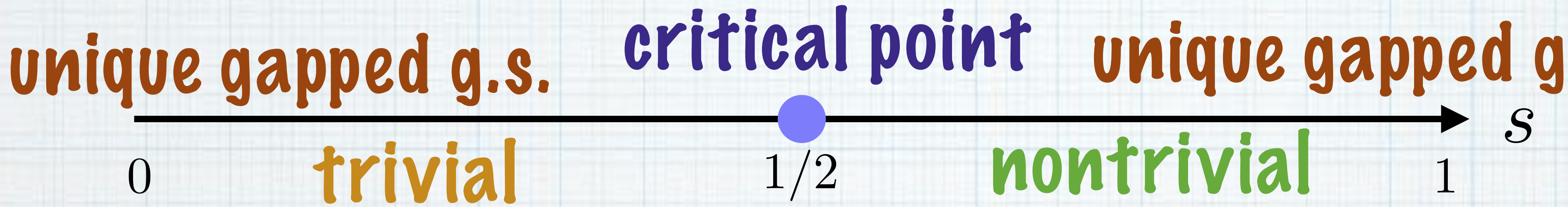
$s = 0$

$$\hat{H}_1^{\mathrm{SSH}} = \sum_j (\hat{c}^\dagger_{2j-1}\hat{c}_{2j} + \mathrm{h.c.}) \qquad |\Phi_{\mathrm{GS},1}\rangle = \Big(\prod_j \frac{\hat{c}^\dagger_{2j-1} - \hat{c}^\dagger_{2j}}{\sqrt{2}}\Big)|\Phi_{\mathrm{vac}}\rangle$$

$s = 1$



Zak phase as a topological index



Zak phase (Berry phase in the Brillouin zone) Zak 1989

$$\nu := \frac{i}{\pi}\int_0^{2\pi} dk \left\langle \boldsymbol{u}^-(k), \frac{d}{dk}\boldsymbol{u}^-(k) \right\rangle = \begin{cases} 0 & s \in [0, \frac{1}{2}) \\ 1 & s \in (\frac{1}{2}, 1] \end{cases}$$

Zak phase and the expectation value of $\hat{U}_{\mathrm{twist}}$

$$\lim_{L\uparrow\infty} \langle \Phi_{\mathrm{GS}} | \hat{U}_{\mathrm{twist}} | \Phi_{\mathrm{GS}} \rangle = e^{i\pi\nu} = \begin{cases} 1 & s \in [0, \frac{1}{2}) \\ -1 & s \in (\frac{1}{2}, 1] \end{cases}$$

the twist (or the flux-insertion) operator

$$\hat{U}_{\mathrm{twist}} = \exp\left[i \sum_{j=1}^{L} \frac{2\pi j}{L} (\hat{n}_{2j} + \hat{n}_{2j+1} - 1) \right]$$

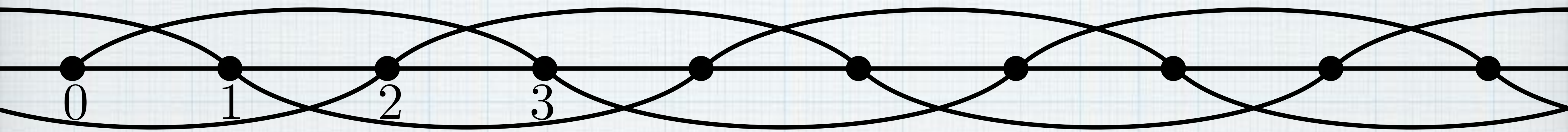
Bloch (Bohm 1949), Lieb, Schultz, Mattis 1961

models
and main results

general model

interacting possibly disordered model of spinless fermions at half-filling with Hamiltonian

$$\hat{H} = \sum_{\substack{j,k\in\mathbb{Z} \\ (j\neq k)}} t_{j,k}\,\hat{c}_j^\dagger \hat{c}_k + \frac{1}{2}\sum_{\substack{j,k\in\mathbb{Z} \\ (j\neq k)}} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$



r_0, t_0, v_0 **constants**

hopping $t_{j,k} = t_{k,j} \in \mathbb{R}$

$t_{j,k} = 0$ **if** $j-k$ **is even or** $|j-k| \geq r_0$

$$\sum_{k(\neq j)} |t_{j,k}|(|k-j|+1)^2 \leq t_0$$

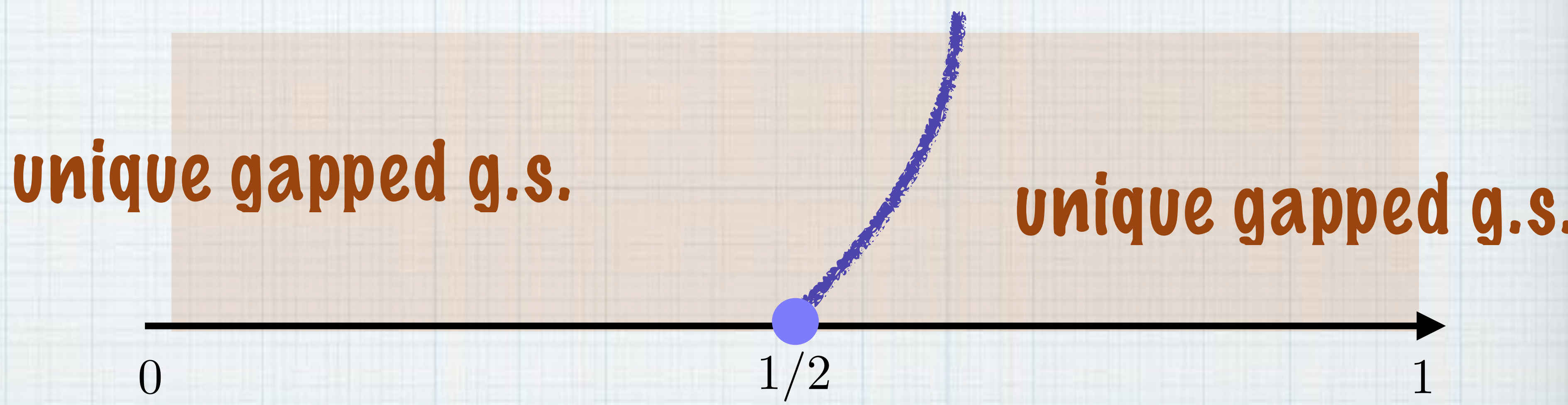
interaction $v_{j,k} = v_{k,j} \in \mathbb{R}$

$v_{j,k} = 0$ **if** $|j-k| \geq r_0$ $\qquad |v_{j,k}| \leq v_0$

an important corollary

$$\hat{H} = \sum_{\substack{j,k\in\mathbb{Z} \\ (j\neq k)}} t_{j,k}\, \hat{c}_j^\dagger \hat{c}_k + \frac{1}{2} \sum_{\substack{j,k\in\mathbb{Z} \\ (j\neq k)}} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$

$\hat{H}_0^{\mathrm{SSH}}$ and $\hat{H}_1^{\mathrm{SSH}}$ belong to different phases within this class of models



$\hat{H}_s$ any path of Hamiltonians (with $s \in [0,1]$) in this class such that $\hat{H}_0 = \hat{H}_0^{\mathrm{SSH}}$ and $\hat{H}_1 = \hat{H}_1^{\mathrm{SSH}}$

$\hat{H}_s$ must go through a phase transition point with either non-unique g.s., gapless g.s., or discontinuity

strategy of the proof

- the model has no translation invariance

  no band structure!

- the model has interactions

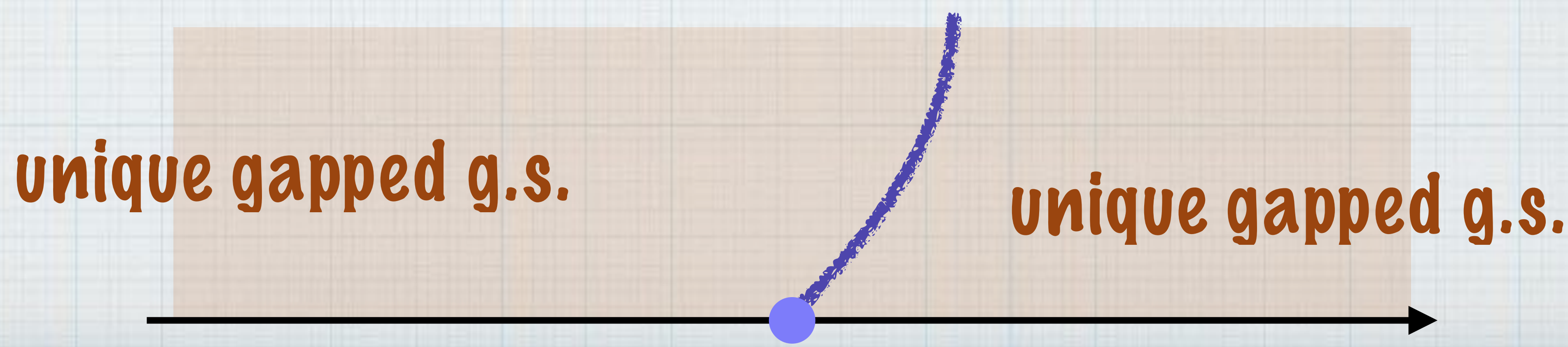
  the ground state is not a Slater determinant, but an intractable many-body state!!

- we shall study phase transitions rigorously

  we must treat infinite systems!!!

we define a $\mathbb{Z}_2$ valued index in terms of the expectation value of the local twist operator in a unique gapped ground state on the infinite chain

Tasaki 2018



symmetry of the models

$$\hat{H} = \sum_{j,k} t_{j,k}\, \hat{c}_j^\dagger \hat{c}_k + \frac{1}{2}\sum_{j,k} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$

- particle number conservation → $U(1)$ symmetry
- invariant under particle-hole transformation + gauge transformation on one of the sublattices

linear *-automorphism Γ 　 $\Gamma(\hat{c}_j) = (-1)^j\, \hat{c}_j^\dagger$

$\Gamma(\hat{n}_j) = 1 - \hat{n}_j$ 　 $\Gamma(\hat{H}) = \hat{H}$ 　 $\Gamma(\hat{A}^\dagger) = \Gamma(\hat{A})^\dagger$, $\Gamma(\hat{A}\hat{B}) = \Gamma(\hat{A})\Gamma(\hat{B})$

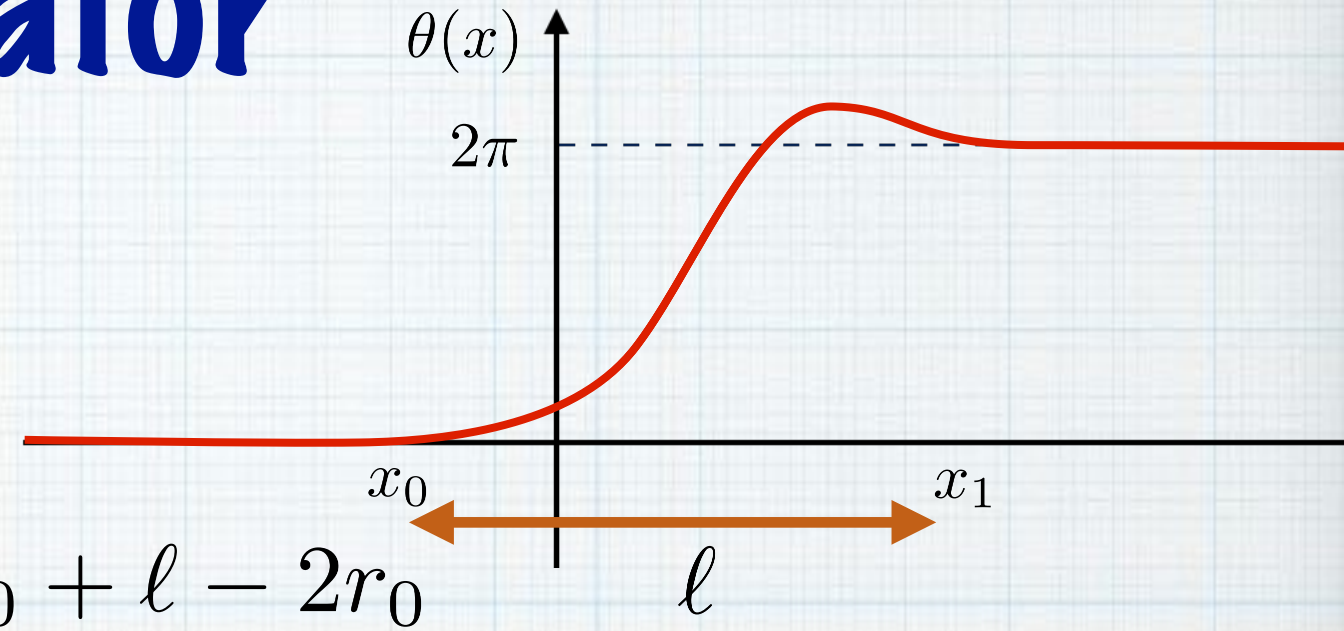
ground state ω on the infinite chain

$|\Phi_{\mathrm{GS}}^{(L)}\rangle$ the ground state on a finite chain $-L/2$ —— $L/2$

infinite volume limit $\omega(\hat{A}) = \lim_{L\uparrow\infty} \langle \Phi_{\mathrm{GS}}^{(L)} | \hat{A} | \Phi_{\mathrm{GS}}^{(L)} \rangle$

unique g.s. is Γ-invariant 　 $\omega(\Gamma(\hat{A})) = \omega(\hat{A})$

general twist operator



function $\theta : \mathbb{R} \to S^1 = [0, 2\pi)$

- $\theta(x) = \begin{cases} 0 & x \le x_0 \\ 2\pi & x \ge x_1 \end{cases}$
- $|\theta'(x)| \le \gamma$
- $\theta(x)$ **wraps around** S^1 **once as** $x : x_0 \to x_1$

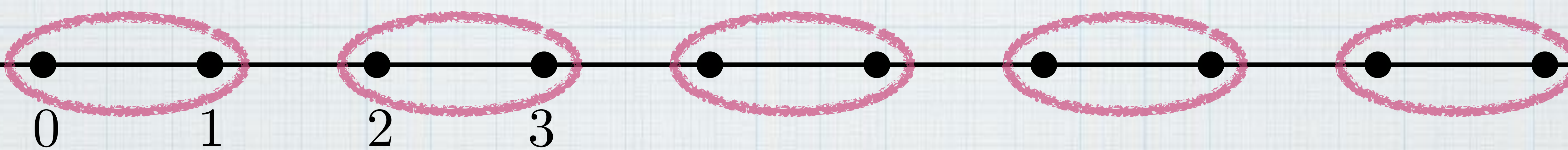
$$x_1 = x_0 + \ell - 2r_0$$

local twist operator

Affleck, Lieb 1986

$$\hat{U}_\theta = \exp\left[i \sum_j \theta(2j)\,(\hat{n}_{2j} + \hat{n}_{2j+1} - 1) \right]$$

$\hat{U}_\theta$ **is local because** $\exp[i\,2\pi\,(\hat{n}_{2j} + \hat{n}_{2j+1} - 1)] = 1$



$\Gamma(\hat{n}_j) = 1 - \hat{n}_j \longrightarrow \Gamma(\hat{U}_\theta) = \hat{U}_\theta^\dagger$

$\omega(\Gamma(\hat{A})) = \omega(\hat{A}) \longrightarrow \omega(\hat{U}_\theta) \in \mathbb{R}$ **reality is essential**

main theorem and the index

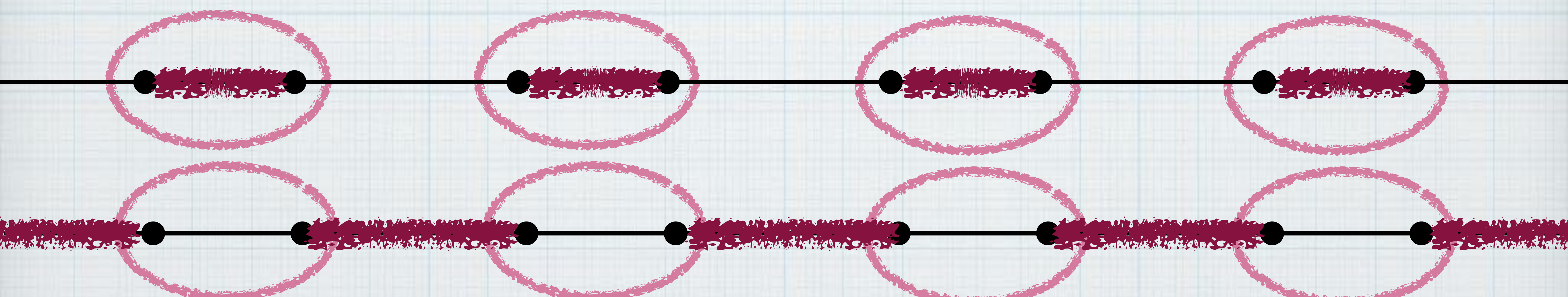
THEOREM: let ω be a unique gapped ground state with energy gap $\Delta E > 0$. for any θ-function with $\gamma^2 \ell < \Delta E / t_0$, $\omega(\hat{U}_\theta)$ is nonzero, and its sign is independent of θ

we define $\mathrm{Ind}_\omega \in \{0,1\} = \mathbb{Z}_2$ by $\mathrm{Ind}_\omega = \begin{cases} 0 & \text{if } \omega(U_\theta) > 0 \\ 1 & \text{if } \omega(U_\theta) < 0 \end{cases}$

trivial (0)

nontrivial (1)

remark: for the two extreme ground state of the SSH model, we recover the Zak phase as $\mathrm{Ind}_{\omega_0} = 0$ and $\mathrm{Ind}_{\omega_1} = 1$



remark: it is believed that $\mathbb{Z}_2$ is the correct classification

invariance of the index

family of Hamiltonians $\hat{H}_s$ with $s \in [0,1]$ (in our class)

- $\hat{H}_s$ has a Γ-invariant unique gapped g.s. ω_s with energy gap $\geq \Delta E_0 > 0$
- $\omega_s(\hat{A})$ is continuous in s for any local operator $\hat{A}$

THEOREM: let ω be a unique gapped ground state with energy gap $\Delta E > 0$. for any θ-function with $\gamma^2 \ell < \Delta E / t_0$, $\omega(\hat{U}_\theta)$ is nonzero, and its sign is independent of θ

COROLLARY: the index Ind_{ω_s} is independent of $s \in [0,1]$

proof: fix a θ-function with $\gamma^2 \ell < \Delta E_0 / t_0$

the theorem implies $\omega_s(\hat{U}_\theta) \neq 0$ for any $s \in [0,1]$

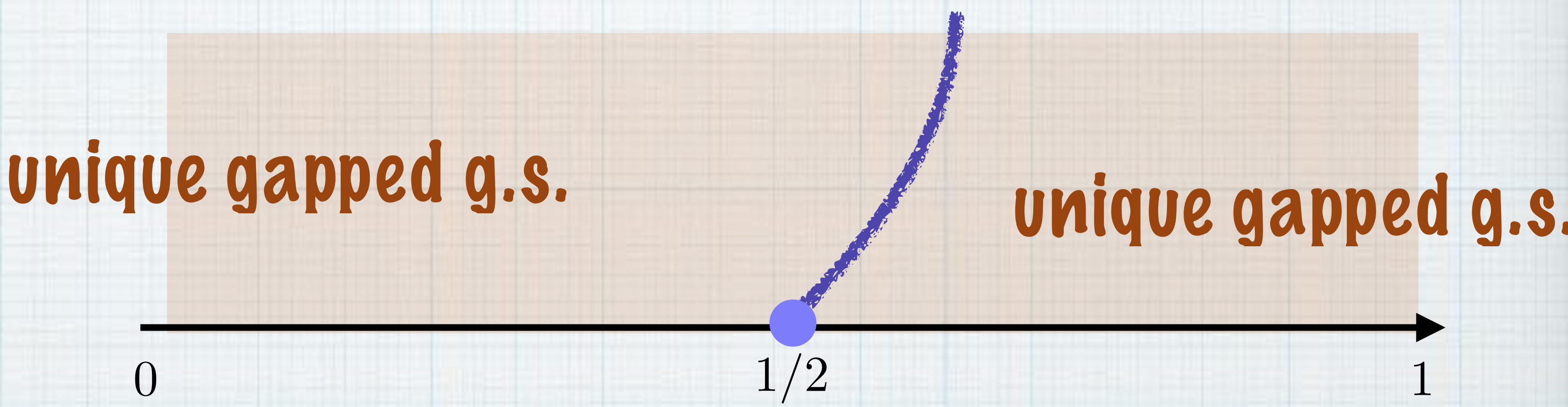
$\omega_s(\hat{U}_\theta)$ cannot change the sign because of continuity

if $\mathrm{Ind}_{\omega_0} \neq \mathrm{Ind}_{\omega_1}$ there must be a phase transition!

an important corollary

$$\hat{H} = \sum_{\substack{j,k\in\mathbb{Z}\\(j\neq k)}} t_{j,k}\,\hat{c}_j^\dagger \hat{c}_k + \frac{1}{2}\sum_{\substack{j,k\in\mathbb{Z}\\(j\neq k)}} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$

$\hat{H}_0^{\mathrm{SSH}}$ and $\hat{H}_1^{\mathrm{SSH}}$ belong to different phases within this class of models



$\hat{H}_s$ any path of Hamiltonians (with $s \in [0,1]$) in this class such that $\hat{H}_0 = \hat{H}_0^{\mathrm{SSH}}$ and $\hat{H}_1 = \hat{H}_1^{\mathrm{SSH}}$

$\hat{H}_s$ must go through a phase transition point with either non-unique g.s., gapless g.s., or discontinuity

proof of the main theorem

a finite chain

proof for a finite chain (periodic b.c.)

THEOREM: let ω be a unique gapped ground state with energy gap $\Delta E > 0$. for any θ-function with $\gamma^2\ell < \Delta E/t_0$, $\omega(\hat{U}_\theta)$ is nonzero, and its sign is independent of θ

- a unique ground state $|\Phi_{\rm GS}\rangle$ with a gap $\Delta E > 0$

$$\omega(\cdot) = \langle\Phi_{\rm GS}|\cdot|\Phi_{\rm GS}\rangle$$

- take a θ-function with $\gamma^2\ell < \Delta E/t_0$
- standard Bloch, Lieb-Schultz-Mattis estimate

$$\langle\Phi_{\rm GS}|\hat{U}_\theta^\dagger \hat{H}\hat{U}_\theta|\Phi_{\rm GS}\rangle - E_{\rm GS} \le t_0\gamma^2\ell < \Delta E$$

- if $\omega(\hat{U}_\theta) = \langle\Phi_{\rm GS}|\hat{U}_\theta|\Phi_{\rm GS}\rangle = 0$, $\hat{U}_\theta|\Phi_{\rm GS}\rangle$ is an excited state with excitation energy $< \Delta E$. so we see $\omega(\hat{U}_\theta) \neq 0$
- since $\omega(\hat{U}_\theta) \in \mathbb{R}$ varies continuously when we modify θ-function continuously, the sign cannot change

proof for a finite chain (periodic b.c.)

THEOR … **ith**
energy … $\Delta E/t_0$,
$\omega(\hat{U}_\theta)$ **is** …

$\hat{H}$ and $|\Phi_{\rm GS}\rangle$ are invariant under
uniform $U(1)$ rotation $e^{i\sum_j \zeta \hat{n}_j}$

non-uniform rotation $U_\theta = (\mathrm{const})e^{i\sum_j \theta_j \hat{n}_j}$
should change the expectation value of $\hat{H}$
only by $\sim (\mathrm{const})(\theta')^2 \times \ell$

- a uniq … $_{\rm GS}| \cdot |\Phi_{\rm GS}\rangle$
- take a θ-function with $\gamma^2\ell < \Delta E/t_0$
- standard Bloch, Lieb-Schultz-Mattis estimate

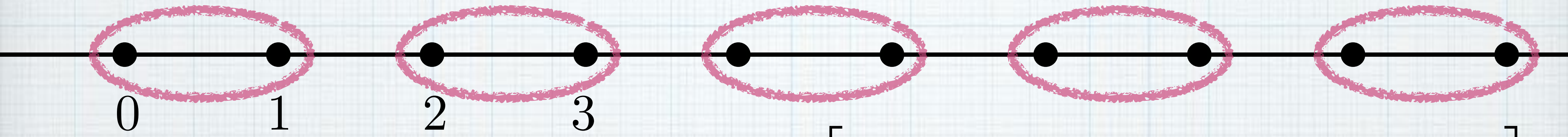
$$\langle\Phi_{\rm GS}|\hat{U}_\theta^\dagger \hat{H} \hat{U}_\theta|\Phi_{\rm GS}\rangle - E_{\rm GS} \le t_0\gamma^2\ell < \Delta E$$

- if $\omega(\hat{U}_\theta) = \langle\Phi_{\rm GS}|\hat{U}_\theta|\Phi_{\rm GS}\rangle = 0$, $\hat{U}_\theta|\Phi_{\rm GS}\rangle$ is an excited state with excitation energy $< \Delta E$. so we see $\omega(\hat{U}_\theta) \neq 0$
- since $\omega(\hat{U}_\theta) \in \mathbb{R}$ varies continuously when we modify θ-function continuously, the sign cannot change

other theorems

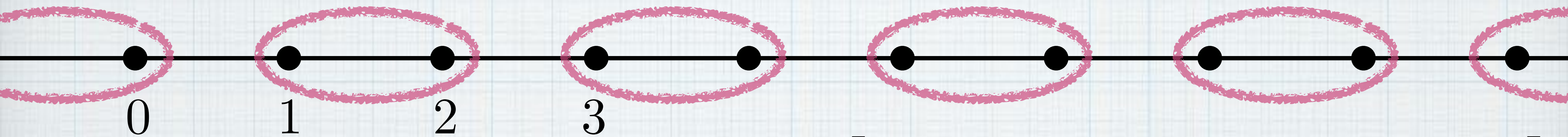
duality of indices

ω **unique gapped ground state**



the twist operator $\hat{U}_\theta = \exp\left[i \sum_j \theta(2j) \left(\hat{n}_{2j} + \hat{n}_{2j+1} - 1 \right) \right]$

defines the index $\mathrm{Ind}_\omega \in \mathbb{Z}_2$



the twist operator $\hat{U}'_\theta = \exp\left[i \sum_j \theta(2j) \left(\hat{n}_{2j-1} + \hat{n}_{2j} - 1 \right) \right]$

defines another index $\mathrm{Ind}'_\omega \in \mathbb{Z}_2$

THEOREM: $\mathrm{Ind}_\omega + \mathrm{Ind}'_\omega = 1$

any unique gapped g.s. is topologically nontrivial either with respect to Ind_ω **or** Ind'_ω

edge mode

$$\hat{H} = \sum_{j,k\in\mathbb{Z}} t_{j,k}\,\hat{c}_j^\dagger\hat{c}_k + \frac{1}{2}\sum_{j,k\in\mathbb{Z}} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$

further assume translation invariance as

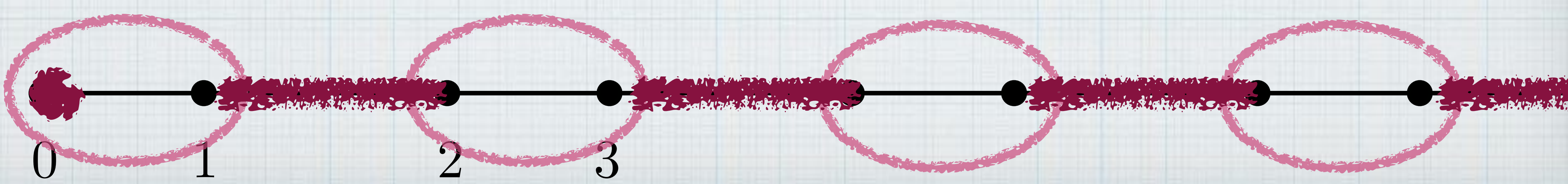
$$v_{j+r_1,k,+r_1} = v_{j,k} \qquad t_{j+r_1,k,+r_1} = t_{j,k} \quad (r_1 \text{ even constant})$$

Hamiltonian on the half-infinite chain $\{0,1,\ldots\}$

$$\hat{H}_+ = \sum_{j,k\geq 0} t_{j,k}\,\hat{c}_j^\dagger\hat{c}_k + \frac{1}{2}\sum_{j,k\geq 0} v_{j,k}(\hat{n}_j - \tfrac{1}{2})(\hat{n}_k - \tfrac{1}{2})$$

THEOREM: suppose that the g.s. ω of $\hat{H}$ is unique (in the global sense), gapped, and satisfies $\mathrm{Ind}_\omega = 1$. Then any Γ-invariant g.s. ω_+ of $\hat{H}_+$ is accompanied by a particle-number-conserving gapless excitation near the edge

for any $\varepsilon > 0$ there is a local unitary $\hat{U}_\varepsilon$ s.t. $\omega_+(\hat{U}_\varepsilon) = 0$ and $\omega_+(\hat{U}_\varepsilon^\dagger[\hat{H},\hat{U}_\varepsilon]) \leq \varepsilon$



summary

- ☑ rigorous but very elementary index theory that applies to a class of interacting one-dimensional topological insulators, including the SSH model
- ☑ our $\mathbb{Z}_2$ index is defined from the sign of the expectation value of the twist operator
- ☑ the index is invariant under continuous modification of unique gapped ground states (with symmetry)
- ☑ a ground state with nontrivial index has a gapless edge excitation when defined on the half-infinite chain

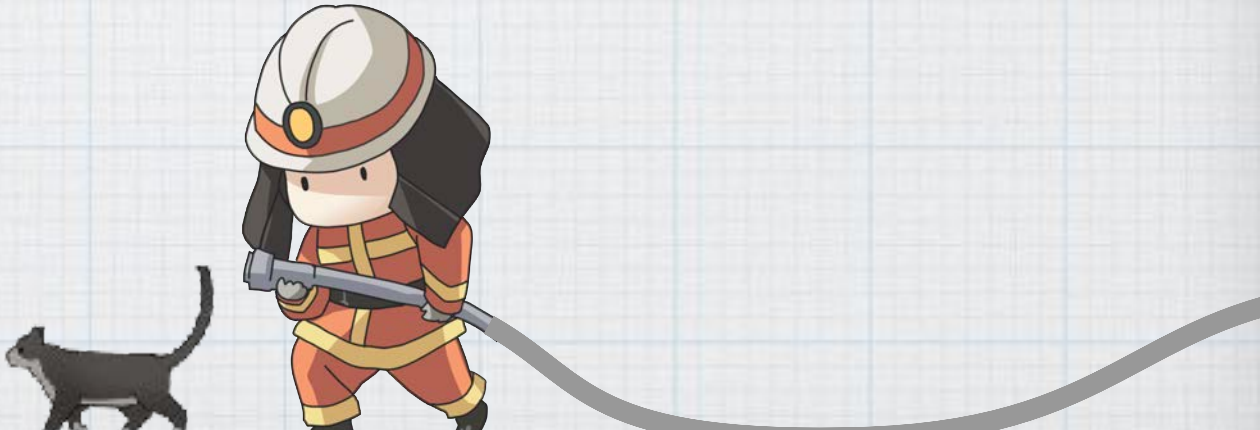


illustration by Chisato Naruse