Griffiths-type theorems for short-range spin glass models

Chigak Itoi, Hisamitsu Mukaida, and Hal Tasaki

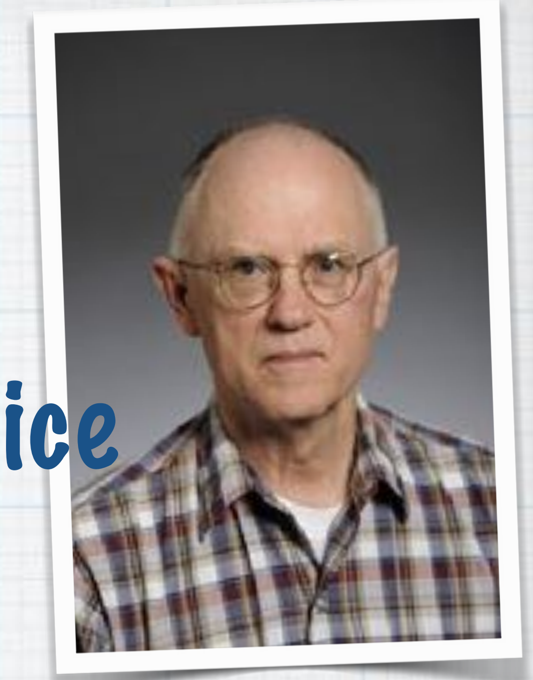
webinar @ YouTube / 2023

introduction

definitions and main results

discussion

Griffiths' theorem for the ferromagnetic Ising model



Ising model on the $L \times \cdots \times L$ hypercubic lattice

Hamiltonian $H_L(\boldsymbol{\sigma}) = -\sum_{\langle x,y \rangle} \sigma_x \sigma_y$

two ways of characterizing the ferromagnetic order

long-range order $\mu_{\mathrm{LRO}} = \lim_{L\uparrow\infty} \sqrt{L^{-2d} \sum_{x,y} \langle \sigma_x \sigma_y \rangle_{L,\beta}} > 0$

correlation $\langle \sigma_x \sigma_y \rangle_\beta$ does not vanish as $|x-y| \uparrow \infty$

spontaneous magnetization $\mu_{\mathrm{SM}} = -\lim_{h\downarrow 0} \frac{\partial f(\beta,h)}{\partial h} > 0$

the free energy is non-differentiable at $h=0$

infinitesimal magnetic field causes nonzero magnetization

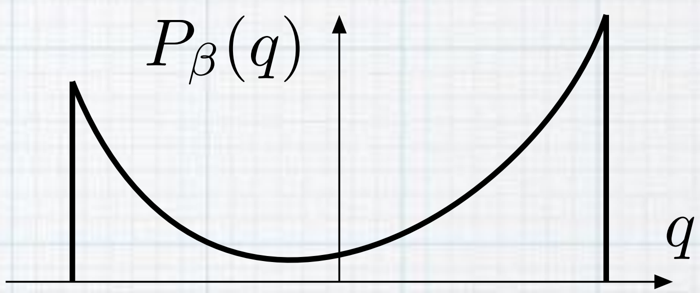
spontaneous symmetry breaking

THEOREM: (Griffiths 1966) $\mu_{\mathrm{SM}} \geq \mu_{\mathrm{LRO}}$

different characterizations of spin glass order

broadening of the overlap distribution

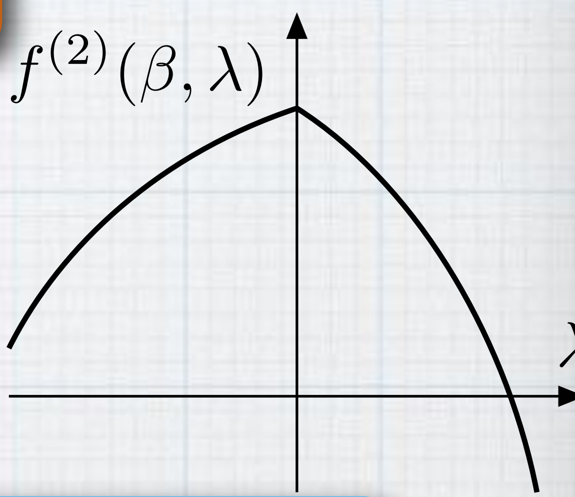
$$q_{\mathrm{br}} = \sqrt{\overline{\langle R^2\rangle} - \left(\overline{\langle R\rangle}\right)^2} > 0$$



non-differentiability of the 2-replica free energy

$$q_{\mathrm{jump}} = -\frac{1}{2}\left\{\lim_{\lambda\downarrow 0}\frac{\partial f^{(2)}(\beta,\lambda)}{\partial\lambda} - \lim_{\lambda\uparrow 0}\frac{\partial f^{(2)}(\beta,\lambda)}{\partial\lambda}\right\}$$

$$= \frac{1}{2}\left\{\overline{\langle R\rangle_+} - \overline{\langle R\rangle_-}\right\} > 0$$



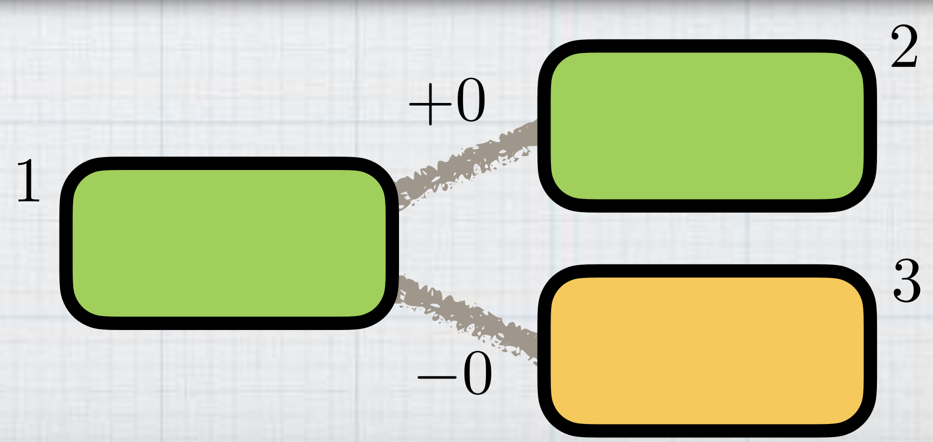
$$q_{\mathrm{jump}} = q_{\mathrm{EA}} = \overline{\langle R\rangle_+} > 0$$

models without magnetic field

literal breakdown of replica symmetry in the 3-replica system

$$q_{\mathrm{rsb}} = -\lim_{\lambda\downarrow 0}\frac{\partial f^{(3)}(\beta,\lambda,-\lambda)}{\partial\lambda}$$

$$= \overline{\langle R^{12}\rangle} - \overline{\langle R^{13}\rangle} > 0$$



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Edwards-Anderson (EA) model with a magnetic field

periodic b.c. (in this video)

Λ_L : d-dimensional $L \times \cdots \times L$ hypercubic lattice

$\sigma_x = \pm 1$: spin on $x \in \Lambda_L$ $\boldsymbol{\sigma} = (\sigma_x)_{x \in \Lambda_L}$: spin configuration

Hamiltonian $H_L(\boldsymbol{\sigma}) = -\sum_{\langle x,y \rangle} J_{xy}\sigma_x\sigma_y - \sum_{x \in \Lambda_L} h_x \sigma_x$

nearest neighbor pairs in Λ_L

$J_{xy} = J_{yx} \in \mathbb{R}$ random interaction (i.i.d.)

$h_x \in \mathbb{R}$ random (or non-random) magnetic field (i.i.d.)

the equilibrium state at β in the limit $L \uparrow \infty$

standard EA model ($h_x = 0$ for all x)

may exhibit a spin glass phase if $d \geq 3$

EA model with a magnetic field

not known if it exhibits a spin glass phase in $d = 3$

it probably has a spin glass phase if $d > 6$

Edwards-Anderson (EA) model with a ma

we do not discuss the existence or the nature of the spin glass phase, but study the relations between different characterizations of "spin glass order"

video)

Λ_L :

$\sigma_x =$ ... ration

Ham... $\sum_{\langle x,y \rangle}$ ... $\sum_{x \in \Lambda_L}$

nearest neighbor pairs in Λ_L

$J_{xy} = J_{yx} \in \mathbb{R}$ random interaction (i.i.d.)

$h_x \in \mathbb{R}$ random (or non-rand ... i.d.)

but the nature of the phase is still controversial!

the equilibrium state at β in the

standard AE model ($h_x = 0$ for a...

this is also controversial!

... a spin glass phase if $d \geq 3$

AE model with a magnetic field

not known if it exhibits a spin glass phase in $d = 3$

it probably has a spin glass phase if $d > 6$

replica overlap and the broadening order parameter $q_{\rm br}$

expectation value for two independent replicas

$$\langle \cdots \rangle^{(2)}_{L,\beta} = \sum_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2} (\cdots) \frac{e^{-\beta H_L(\boldsymbol{\sigma}^1)}}{Z_L(\beta)} \frac{e^{-\beta H_L(\boldsymbol{\sigma}^2)}}{Z_L(\beta)}$$

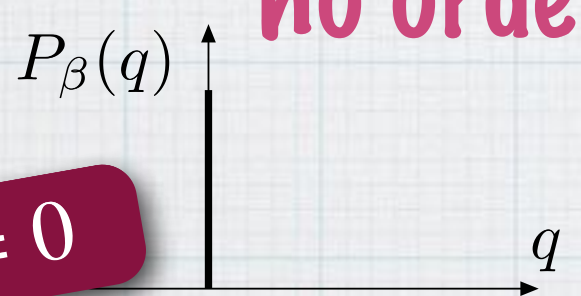
$$\boldsymbol{\sigma}^1 = (\sigma^1_x)_{x\in\Lambda_L}$$
$$\boldsymbol{\sigma}^2 = (\sigma^2_x)_{x\in\Lambda_L}$$

replica overlap $R = L^{-d} \sum_x \sigma^1_x \sigma^2_x$

random average

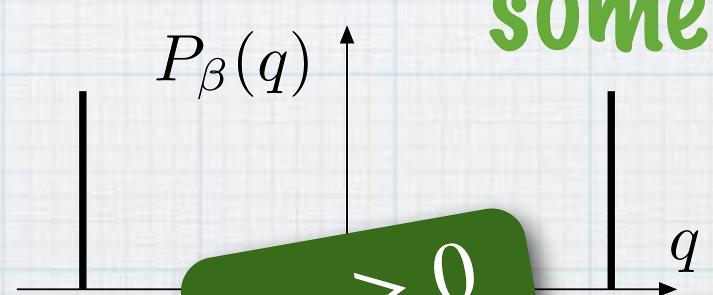
replica overlap distribution $P_\beta(q) = \lim_{L\uparrow\infty} \overline{\langle \delta(R-q)\rangle^{(2)}_{L,\beta}}$

no order

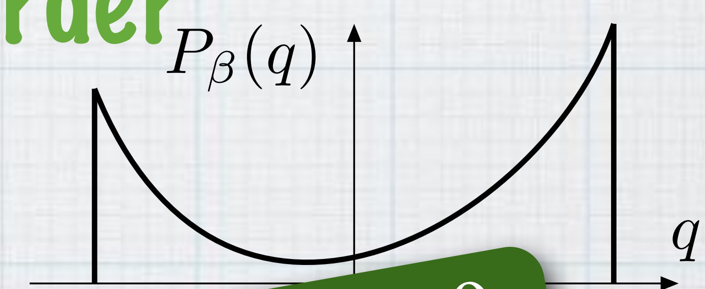


$q_{\rm br} = 0$

some order



$q_{\rm br} > 0$



$q_{\rm br} > 0$

order = broadening of $P_\beta(q)$ can be quantified by

$$q_{\rm br} = \sqrt{\int dq\, q^2\, P_\beta(q) - \left\{\int dq\, q\, P_\beta(q)\right\}^2} = \lim_{L\uparrow\infty} \sqrt{\overline{\langle R^2\rangle^{(2)}_{L,\beta}} - \left(\overline{\langle R\rangle^{(2)}_{L,\beta}}\right)^2}$$

non-differentiability of the 2-replica free energy and the jump order parameter q_{jump}

two replicas with explicit coupling $\lambda \in \mathbb{R}$

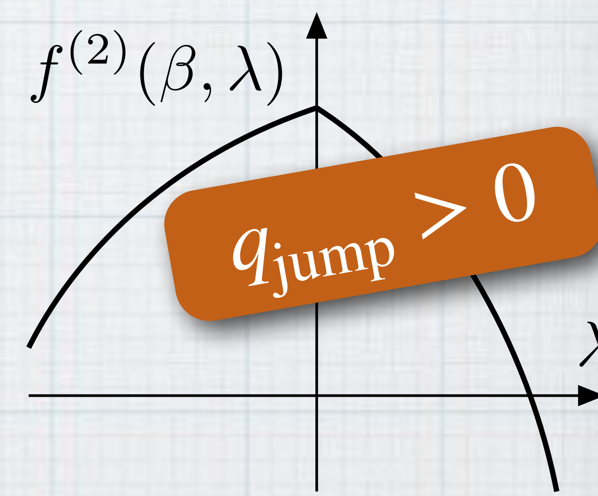
Hamiltonian $H_L(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2; \lambda) = H_L(\boldsymbol{\sigma}^1) + H_L(\boldsymbol{\sigma}^2) - \lambda\, \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2$

free energy $f^{(2)}(\beta, \lambda) = -\lim_{L\uparrow\infty} \frac{1}{\beta L^d} \overline{\log \sum_{\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2} e^{-\beta H_L(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2; \lambda)}}$

if the system has any order, $f^{(2)}$ should be singular at $\lambda = 0$

$$q_{\text{jump}} = -\tfrac{1}{2}\left\{\lim_{\lambda\downarrow 0} \frac{\partial f^{(2)}(\beta,\lambda)}{\partial \lambda} - \lim_{\lambda\uparrow 0} \frac{\partial f^{(2)}(\beta,\lambda)}{\partial \lambda}\right\}$$

$$= \tfrac{1}{2}\left\{\overline{\langle R\rangle^{(2)}_{\beta,\lambda=+0}} - \overline{\langle R\rangle^{(2)}_{\beta,\lambda=-0}}\right\}$$



van Enter, Griffiths 1983

$q_{\text{jump}} > 0 \iff f^{(2)}$ **is non-differentiable at $\lambda = 0$**

for the standard EA model without a magnetic field

$$q_{\text{jump}} = q_{\text{EA}} = \overline{\langle R\rangle^{(2)}_{\beta,\lambda=+0}}$$

EA order parameter

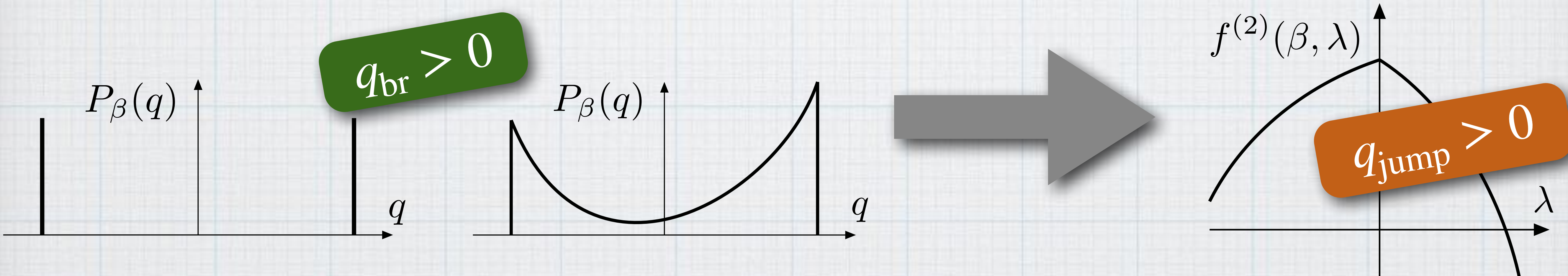
the first and the second theorems

THEOREM 1: in the standard EA model without a magnetic field, one has $q_{\mathrm{br}} \leq q_{\mathrm{EA}} = q_{\mathrm{jump}}$

straight forward extension of Griffiths' theorem

THEOREM 2: in the general EA model, one has $\dfrac{(q_{\mathrm{br}})^2}{4} \leq q_{\mathrm{jump}}$

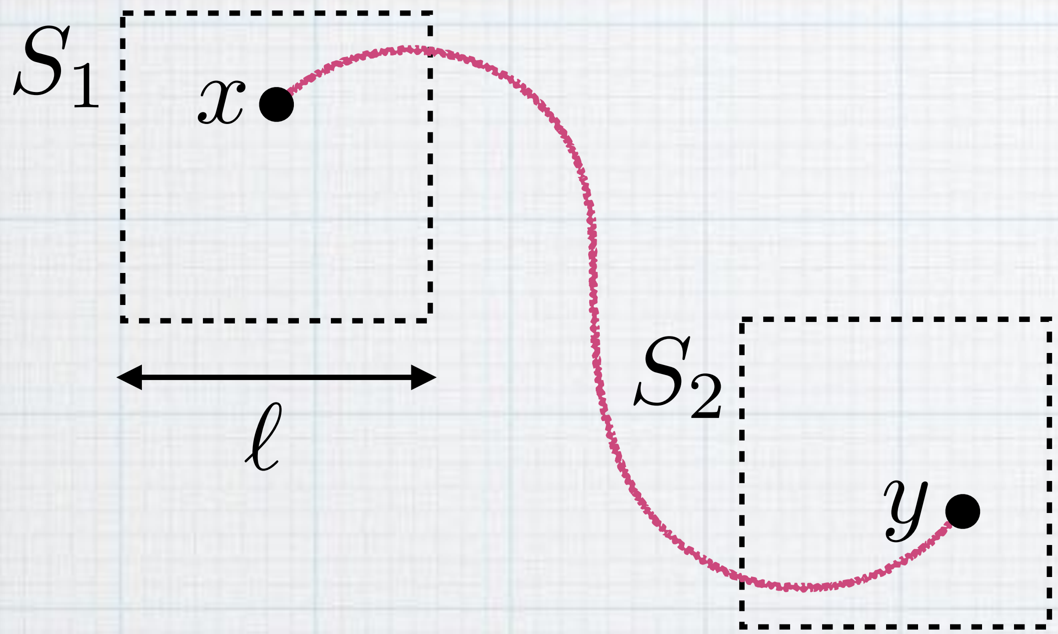
broadening of the overlap distribution implies non-differentiability of the two-replica free energy

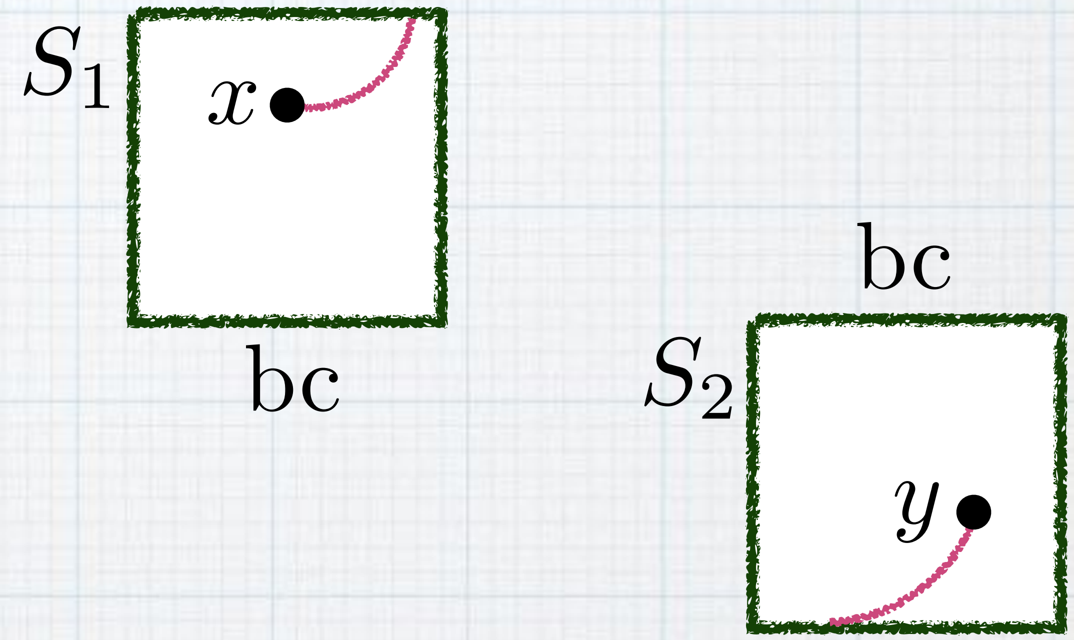


the theorems can be proved for any classical spin glass model with short-range interaction and bounded spins

the idea of the proof (for the EA model)

$$q_{\rm br} = \lim_{L\uparrow\infty} \sqrt{\overline{\langle R^2\rangle^{(2)}_{L,\beta}}} = \lim_{L\uparrow\infty} L^{-d}\sqrt{\overline{\textstyle\sum_{x,y\in\Lambda_L}\left(\langle\sigma_x\sigma_y\rangle_{L,\beta}\right)^2}}$$





$$\sum_{\substack{x\in S_1\\ y\in S_2}} \left(\langle\sigma_x\sigma_y\rangle_{L,\beta}\right)^2 \le \max_{\rm bc} \sum_{x\in S_1}\left(\langle\sigma_x\rangle_{\ell,\beta,{\rm bc}}\right)^2 \max_{\rm bc}\sum_{y\in S_2}\left(\langle\sigma_y\rangle_{\ell,\beta,{\rm bc}}\right)^2$$

Tasaki 1989

Edwards-Anderson order parameter

$$q_{\rm EA} := \lim_{\ell\uparrow\infty}\frac{1}{\ell^d}\,\overline{\max_{\rm bc}\sum_{x\in\Lambda_\ell}\left(\langle\sigma_x\rangle_{\ell,\beta,{\rm bc}}\right)^2} = -\lim_{\lambda\downarrow 0}\frac{\partial f^{(2)}(\beta,\lambda)}{\lambda}$$

van Enter, Griffiths 1983

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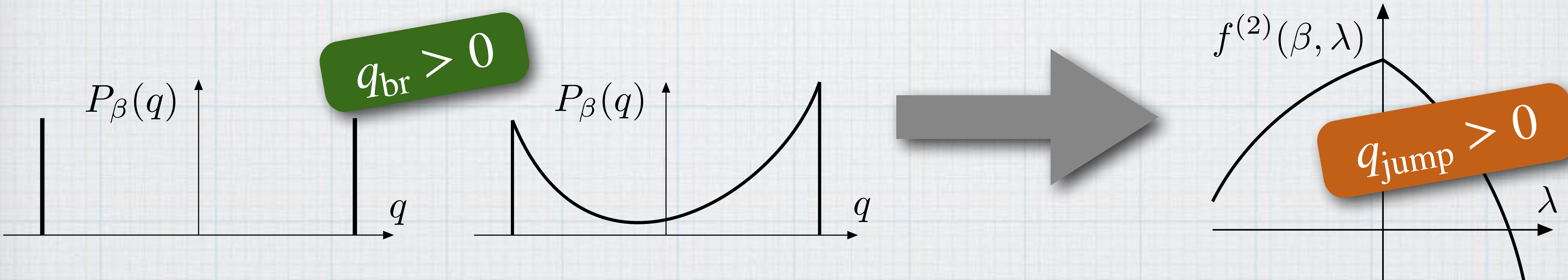
straight forward extension of Griffiths' theorem

THEOREM 2: in the general EA model, one has $\frac{(q_{\mathrm{br}})^2}{4} \leq q_{\mathrm{jump}}$

broadening of the overlap distribution implies non-differentiability of the two-replica free energy



the theorems can be proved for any classical spin glass model with short-range interaction and bounded spins

3-replica free energy and the literal replica symmetry breaking (RSB) order parameter $q_{\rm rsb}$

Guerra 2013

three replicas with explicit couplings $\lambda, \lambda' \in \mathbb{R}$

Hamiltonian

$$H_L(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3; \lambda, \lambda') = H_L(\boldsymbol{\sigma}^1) + H_L(\boldsymbol{\sigma}^2) + H_L(\boldsymbol{\sigma}^3) - \lambda\, \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 - \lambda'\, \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^3$$

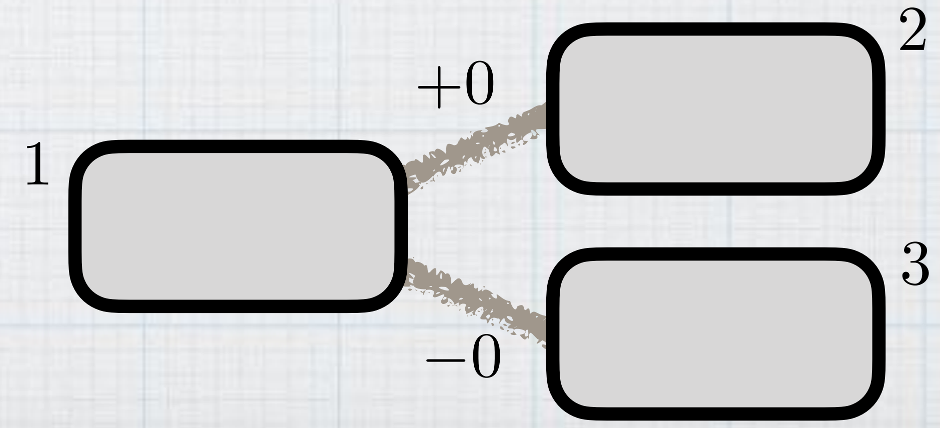
free energy

$$f^{(3)}(\beta, \lambda, \lambda') = -\lim_{L\uparrow\infty} \frac{1}{\beta L^d} \overline{\log \sum_{\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3} e^{-\beta H_L(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3; \lambda, \lambda')}}$$

the measure of spontaneous breakdown of the permutation symmetry of the three replicas

$$q_{\rm rsb} = -\lim_{\lambda\downarrow 0} \frac{\partial f^{(3)}(\beta, \lambda, -\lambda)}{\partial \lambda}$$

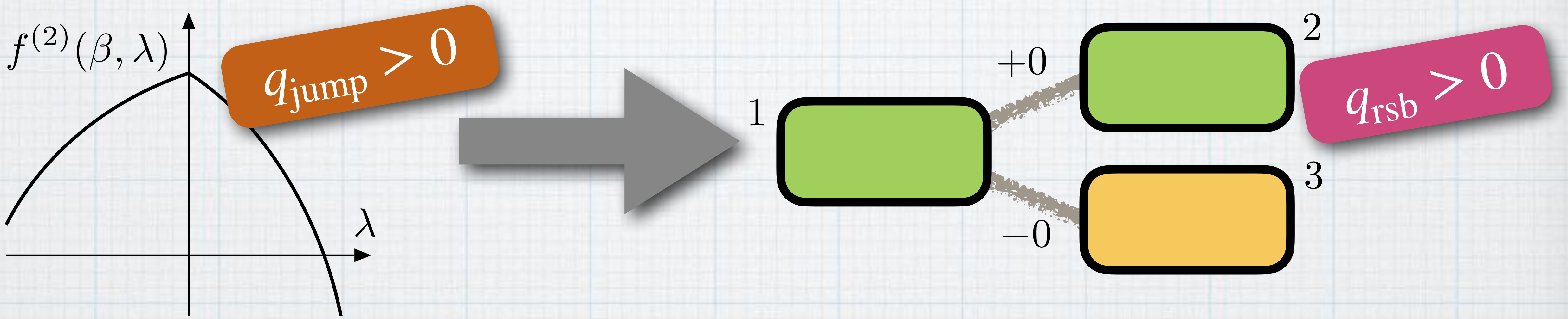
$$= \overline{\langle R^{12} \rangle} - \overline{\langle R^{13} \rangle}$$



the third theorem

THEOREM 3: in almost any spin glass model (including long-range models), one has $2\,q_{\text{jump}} \le q_{\text{rsb}}$

non-differentiability of the two-replica free energy implies literal replica symmetry breaking



that $q_{\text{jump}} > 0$ has been proved (only) in long-range spin glass models such as the Sherrington-Kirkpatrick (SK) model (with or without a magnetic field) and the random energy model

Talagrand 2003, 2011, Guerra 2013

Theorem 3 establishes that these models exhibit literal replica symmetry breaking

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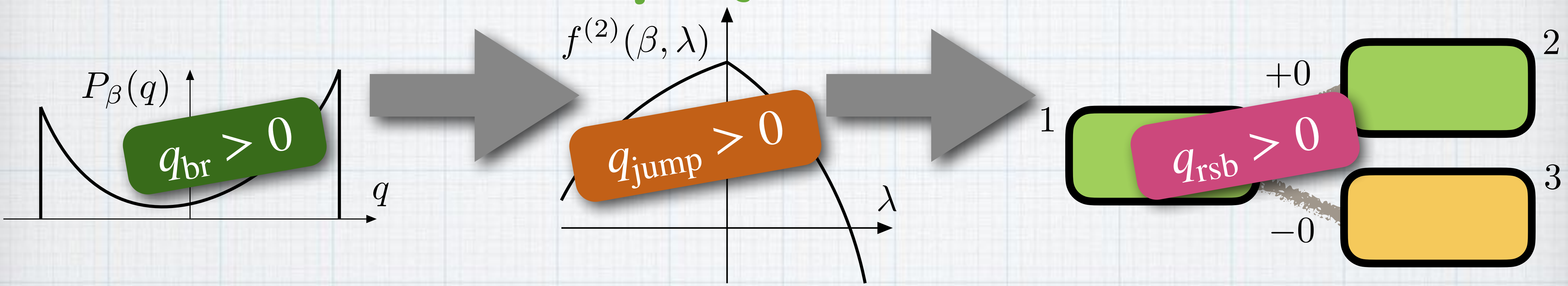
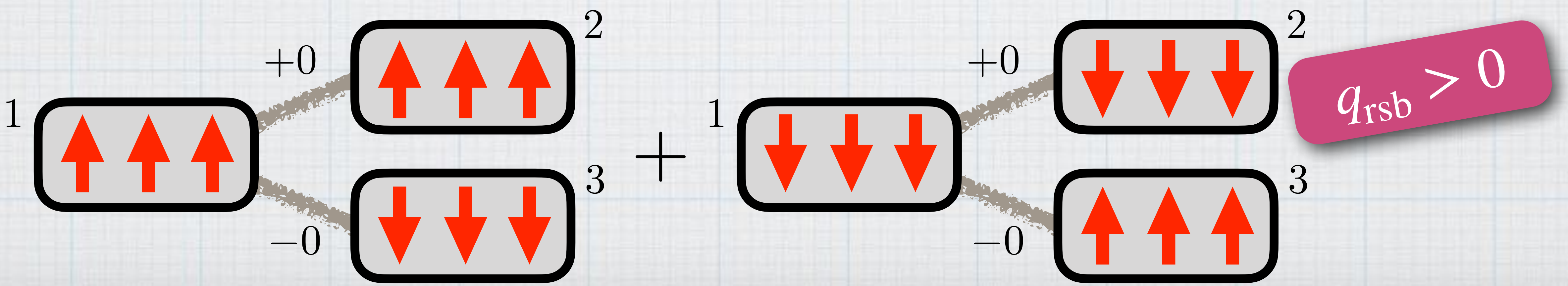
limitation of the theory

we established relations between the different characterizations of "spin glass order"



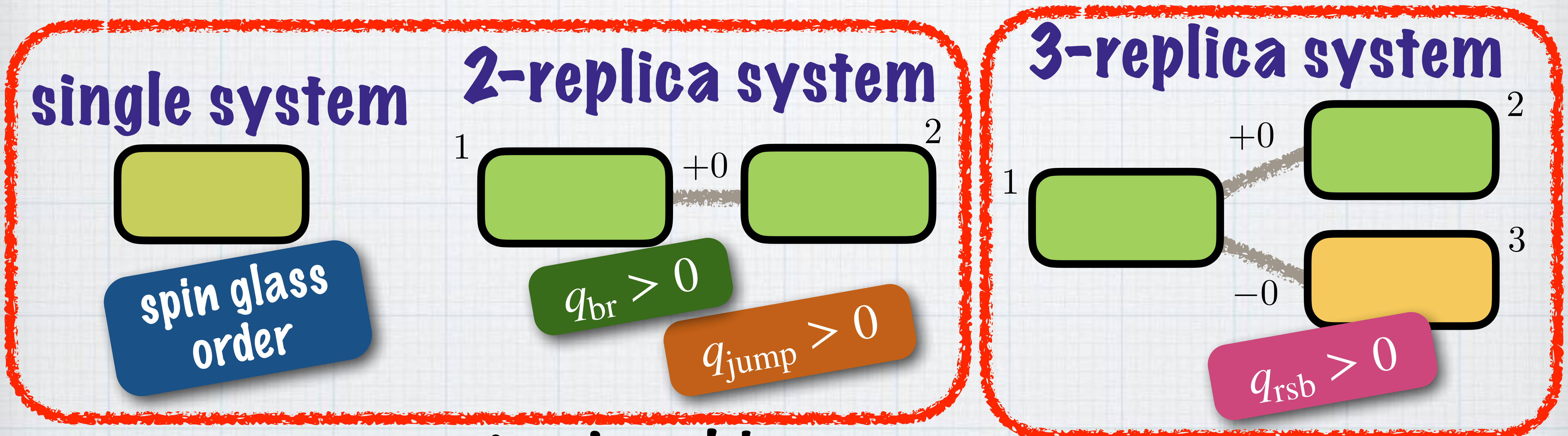
none of the conditions have been proved rigorously for short-range spin glass models

these conditions do not always imply spin glass order.
in the ferromagnetic Ising model, $\mu_{\mathrm{SM}} > 0$ implies $q_{\mathrm{br}} > 0$, $q_{\mathrm{jump}} > 0$, and $q_{\mathrm{rsb}} > 0$



spin glass models under a magnetic field

our theory is probably most meaningful for (short-range) spin glass models under a (random or non-random) magnetic field, which have no obvious symmetry



no symmetry breaking (in the standard sense)

manifest spontaneous symmetry breaking

it may be that there is no spin glass order in $d = 3$

Sasaki, Hukushima, Yoshino, Takayama 2007, Baity-Jesi et al. 2014

summary

- ☑ we proved inequalities for general short-range spin glass models that clarify the relations between different characterizations of "spin glass order"
- ☑ we proved that $q_{\mathrm{br}} > 0$ implies $q_{\mathrm{jump}} > 0$, and then $q_{\mathrm{jump}} > 0$ implies $q_{\mathrm{rsb}} > 0$
- ☑ the theory has an interesting implication in a model with a magnetic field. such a model does not exhibit a symmetry breaking by itself but may exhibit spontaneous breakdown of replica permutation symmetry in the 3-replica system
- ☑ our inequality may be used to prove the absence of spin glass order in some short-range models