$S=\frac{1}{2}$ XX

The ~~XY and XY~~Z models on the two ~~or higher~~ dimensional hypercubic lattice do not possess nontrivial local conserved quantities

sketch of the proof

Naoto Shiraishi and Hal Tasaki

webinar@YouTube, 2024

<model and the main result>

$\Lambda = \{1, \dots, L\}^2$ (1) square lattice (periodic b.c.)

$\mathcal{B} = \{\{u,v\} \mid u, v \in \Lambda,\ |u-v| = 1\}$ (2) the set of n.n. pairs

$\hat{X}_u, \hat{Y}_u, \hat{Z}_u$ copies of the Pauli matrices on site $u \in \Lambda$

Hamiltonian the XX model $S = \frac{1}{2}$

$$\hat{H} = -\sum_{\{u,v\} \in \mathcal{B}} \{\hat{X}_u \hat{X}_v + \hat{Y}_u \hat{Y}_v\} \quad (3)$$

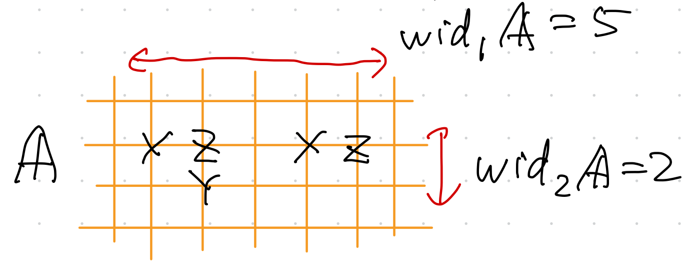
product $\boxed{A = \bigotimes_{u \in S} \hat{A}_u}$ (4) $\Lambda \supset S \neq \emptyset$, $\hat{A}_u \in \{\hat{X}_u, \hat{Y}_u, \hat{Z}_u\}$

the set of all products: $\mathcal{P}$

support of A : $\mathrm{supp}\, A = S$

widths of A : $\mathrm{wid}_\alpha A$ (the width in the α-direction)

$\mathrm{wid}\, A = \max\{\mathrm{wid}_1 A, \mathrm{wid}_2 A\}$



local conserved quantity

$k_{max} = 3, 4, \cdots, \frac{L}{2}$

$$\boxed{\hat{Q} = \sum_{\substack{A \in \mathcal{P} \\ (\mathrm{wid}\, A \le k_{max})}}' q_A\, A,} \qquad (1)$$

$q_A \in \mathbb{C}$, $q_A \neq 0$ for some A with $\mathrm{wid}\, A = k_{max}$

no symmetries for q_A are assumed.

$$[\hat{H}, \hat{Q}] = 0 \qquad (2)$$

Theorem there is no such $\hat{Q}$

basic strategy of the proof

for general $\hat{Q}$ of the form (1), $[\hat{H}, \hat{Q}] = \sum_{B \in \mathcal{P}}' c_B\, B$ (3)

linear combination of q_A

condition (2) $\Longleftrightarrow$ $\boxed{c_B = 0 \text{ for all } B \in \mathcal{P}}$ $\rightarrow$ coupled linear equations for q_A's

$\Downarrow$

$q_A = 0$ for all A s.t. $\mathrm{wid}\, A = k_{max}$ $\Rightarrow$ contradiction.

<basic relations>

$h \in P$ part of $\hat{H}$ $\quad h = \hat{X}_u \hat{X}_v$ or $\hat{Y}_u \hat{Y}_v$, $\{u, v\} \in B$

$A \in P$, $[h, A] = 0$ (1) or $[h, A] = \pm 2i B$ with $B \in P$ (2)

(2) with some $h \in P$ $\Rightarrow$ $\boxed{B \text{ is generated by } A}$

$B \in P$, $A_1, \dots, A_n$: all products with $\mathrm{wid}\, A_j \leq k_{max}$ that generate B

$$C_B = 2i \sum_{j=1}^{n} \pm q_{A_j} \quad (3) \qquad \therefore \boxed{\sum_{j=1}^{n} \pm q_{A_j} = 0} \quad (4)$$

$\boxed{n=1}$ ⇝ if $\exists B$ s.t. A is the only product with wid $\leq k_{max}$ that generates B

$$q_A = 0 \quad (5)$$

$\boxed{n=2}$ ⇝ if $\exists B$ s.t. A, A' are the only products with wid $\leq k_{max}$ that generate B

$$q_A = \pm q_{A'} \quad (6)$$

<commutation relations — appending operation>

example 1 → important! $\quad [\hat{X}, \hat{Y}] = 2i\hat{Z}$

$$\Big[\underbrace{\overset{u}{X}\,\overset{v}{X}}_{h},\ \overset{u}{\ }\,\overset{v}{Y}\underbrace{Z\,\overset{X}{Z}}_{A}\Big] = 2i\,\underbrace{\overset{u}{X}\,\overset{v}{Z}\,Z\,\overset{X}{Z}}_{B} \qquad (1)$$

→ $\operatorname{supp} B \supsetneq \operatorname{supp} A$

$\{u, v\} \in B$

$$B = \mathcal{A}^{XX}_{u\to v}(A)$$

(B is obtained by appending X at u)

example 2 $\quad [\hat{Y}, \hat{X}] = -2i\hat{Z}, \quad \hat{Y}^2 = \hat{I}$

$$\Big[\overset{u}{Y}\,\overset{v}{Y},\ \overset{u}{X}\,\overset{v}{Y}\,\overset{Z}{X}\Big] = -2i\,\overset{u}{Z}\,\overset{v}{\ }\,\overset{Z}{X} \qquad (2)$$

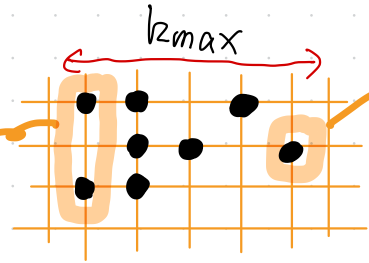
$\operatorname{supp} B \subsetneq \operatorname{supp} A$

$$\Big(\ \Big[\overset{u}{X}\,\overset{v}{X},\ \overset{u}{Z}\,\overset{v}{Y}\,\overset{Z}{X}\Big] = 0, \quad (3) \qquad \Big[\overset{u}{X}\,\overset{v}{X},\ \overset{u}{X}\,\overset{v}{X}\,\overset{Z}{Y}\Big] = 0 \quad (4)\ \Big)$$

⟨monogamy lemma⟩

$A \in P$, $\mathrm{wid}\, A = k_{max}$ $\xrightarrow{\text{without loss of generality}}$ $\mathrm{wid}_1 A = k_{max}$

supp A



left-most sites

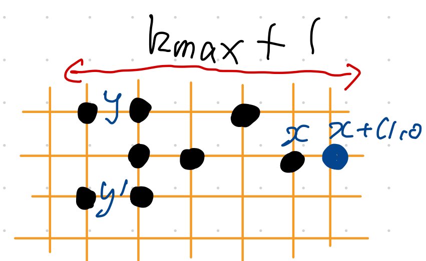
right-most site(s)

lemma if $A \in P$ with $\mathrm{wid}_1 A = k_{max}$ has non-unique left-most sites or right-most sites, then $q_A = 0$

proof x right-most site, y, y' left-most sites

$$B = \mathcal{A}^{ww}_{x+(1,0) \leadsto x}(A)$$

supp B



A is the only product with $\mathrm{wid}_1 \leq k_{max}$ that generates B. $\xrightarrow[(5)]{P3}$ $q_A = 0$ //

⟨Shiraishi-shift⟩

$A \ni P$, $\mathrm{wid}_1 A = k_{max}$, A has a unique $\begin{cases} \text{left-most site } y \\ \text{right-most site } x \end{cases}$

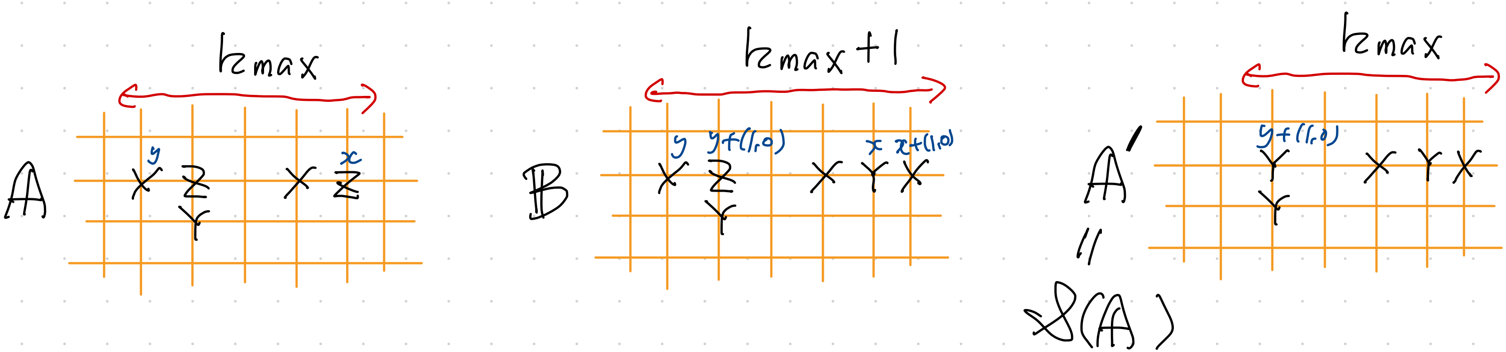
$$B = \Delta^{WW}_{x+(1,0)\to x}(A) \qquad \hat{W} = \begin{cases} \hat{X} & \text{if } \hat{A}_x = \hat{Y} \text{ or } \hat{Z} \\ \hat{Y} & \text{if } \hat{A}_x = \hat{X} \end{cases}$$

Shiraishi-shift of A = $\boxed{\delta(A)} = A'$

if A' s.t. $B = \Delta^{W'W'}_{y\to y+(1,0)}(A')$ with $\hat{W}' = \hat{X}$ or $\hat{Y}$ exists

(if A' exists, then it is unique and has $\mathrm{wid}_1 A' = k_{max}$)

if there is no such $A' \in P$, then the shift $\delta(A)$ does not exist.



7

if the shift $A' = \delta(A)$ exists, A, A' are the only products with $\mathrm{wid}_1 \leq k_{max}$ that generate $\mathbb{B}$

$$p3\text{-}(6) \Rightarrow e_A = \pm e_{\delta(A)} \qquad (1)$$

if shift $\delta(A)$ does not exist, A is the only product with $\mathrm{wid}_1 \leq k_{max}$ that generates $\mathbb{B}$

$$p3\text{-}(5) \Rightarrow e_A = 0 \qquad (2)$$

using (1) repeatedly

$$e_A = \pm e_{\delta^n(A)} \qquad (3) \qquad\qquad \delta^n(A) = \underbrace{\delta \circ \cdots \circ \delta}_{n}(A) \qquad (4)$$

with (2), we get

if $\delta^n(A)$ does not exist, then $e_A = 0$

lemma $A \in \mathcal{P}$, $\mathrm{wid}_1 A = k_{max}$

$\delta^n(A)$ with $n \geq k_{max}$ exists if and only if A is a product of k_{max} Pauli matrices on a straight line of the form

$$A = \hat{W}\,\hat{Z}\,\hat{Z} \cdots \hat{Z}\,\hat{W}' \quad \text{with } \hat{W}, \hat{W}' = \hat{X}, \hat{Y}$$

rough proof the case $\delta^n(A)$ exists for all n

$k_{max} = 5$

A $X\,Z\,Z\,Z\,X$

$X\,Z\,Z\,Z\,Z\,Y$ YY

XX

$\delta(A)$ $Y\,Z\,Z\,Z\,Y$

XX

$Y\,Z\,Z\,Z\,Z\,X$

YY

$\delta^2(A)$ $X\,Z\,Z\,Z\,X$

$\vdots$

rough proof (continued)

$k_{max} = 5$

Y

A X Z Z Z Y

XX

Y

X Z Z Z Z X

XX

Y

$\delta(A)$ Y Z Z Z X

YY

Y

Y Z Z Z Z Y

two left-most sites

$\delta^2(A)$ does not exist

$k_{max} = 5$

A X Z $\sqcup$ Z Y

XX

X Z $\sqcup$ Z Z X

XX

$\delta(A)$ Y $\sqcup$ Z Z X

YY

Y $\sqcup$ Z Z Z Y

$\delta^2(A)$ does not exist

rough proof (continued)

A X Z X Y X   YY

X Z X Y Z Y   XX

$\delta(A)$ Y X Y Z Y   XX

Y X Y Z Z X

$\delta^2(A)$ Z Y Z Z X

Z Y Z Z Z Y   ?

$\delta^3(A)$ does not exist

you can't have Z at the left end

$\Downarrow$

you can't have X or Y except at the two ends

$\Downarrow$

you can't have Z at the right end

$\Downarrow$

you can only have $\hat{X}$ or $\hat{Y}$

$\hat{X}$ or $\hat{Y}$ $\rightarrow$ $O \; \hat{Z} \; \hat{Z} \; \cdots \; \hat{Z} \; \hat{Z} \; O$

//

lemma $A \in P$, $\mathrm{wid}_1 A = k_{max}$
$\gamma^n(A)$ with $n \geq k_{max}$ exists if and only if A is a product of k_{max} Pauli matrices on a straight line of the form

$$A = \hat{W}\hat{Z}\hat{Z}\cdots\hat{Z}\hat{W}' \quad \text{with } \hat{W}, \hat{W}' = \hat{X}, \hat{Y} \qquad (1)$$

if $\gamma^n(A)$ does not exist then $q_A = 0$ $\rightarrow$ P7

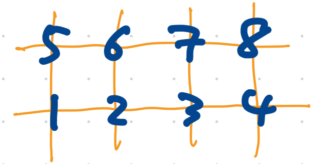
- $A \in P$, $\mathrm{wid}_1 A = k_{max}$, $q_A = 0$ unless A is of the form (1)
- we shall prove $q_{\underbrace{\hat{W}\hat{Z}\cdots\hat{Z}\hat{W}'}_{k_{max}}} = 0$ for $\hat{W}, \hat{W}' = \hat{X}, \hat{Y}$

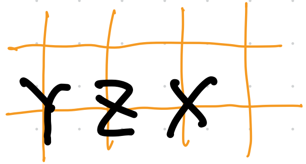
$\Downarrow$

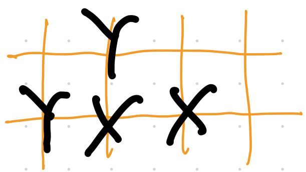
$q_A = 0$ for $^\forall A \in P$ s.t. $\mathrm{wid}_1 A = k_{max}$ $\Rightarrow$ contradiction.

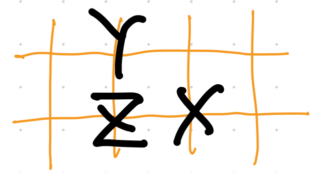
theorem is proved

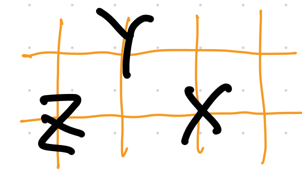
$\langle$ the case with $k_{max} = 3$ $\rangle$



$\mathbb{C}_1 =$ 

$\mathbb{D}_1 =$ 

$\mathbb{E}_2 =$ 

$\mathbb{C}_1'' =$ 

$$\mathbb{D}_1 = \mathcal{A}^{YY\to Z}_{6\to 2}(\mathbb{C}_1) = \mathcal{A}^{YY\to Z}_{1\to 2}(\mathbb{E}_2) = \mathcal{A}^{XX\to Z}_{2\to 1}(\mathbb{C}_1'') \tag{1}$$

$\mathbb{D}_1$ is generated by $\mathbb{C}_1$, $\mathbb{E}_2$, $\mathbb{C}_1''$, and other products with larger support than $\mathbb{D}_1$

$$\hat{Q} = \sum_{\substack{\mathbb{A}\in\mathcal{P} \\ (\mathrm{wid}\,\mathbb{A} \le k_{max})}}' q_{\mathbb{A}}\, \mathbb{A} \qquad (1)$$

$$[\hat{H}, \hat{Q}] = \sum_{\mathbb{B}\in\mathcal{P}}' c_{\mathbb{B}}\, \mathbb{B} \qquad (2)$$

larger support than $\mathbb{D}_1$

$$c_{\mathbb{D}_1} = 2i\left\{ q_{\mathbb{C}_1} + q_{\mathbb{E}_2} - q_{\mathbb{C}_1''} \pm q_{\mathbb{O}} \pm q_{\mathbb{D}} \cdots \right\} \qquad (3)$$

$\mathrm{wid}_1 \geq k_{max}$, $\mathrm{wid}_2 \geq 2$.

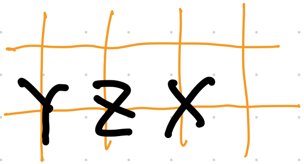
$$\boxed{q = 0}$$

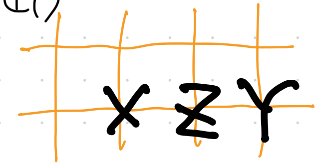
$$c_{\mathbb{D}_1} = 0 \qquad (4)$$

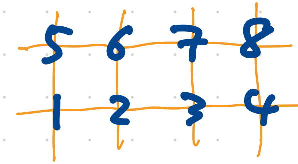
$$\Downarrow$$

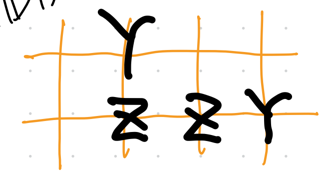
$$q_{\mathbb{C}_1} + q_{\mathbb{E}_2} = 0 \qquad (5)$$

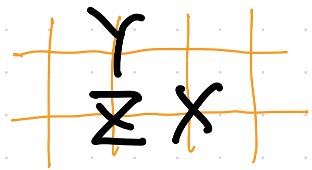
14

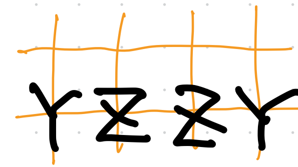
$\mathbb{C}_1 =$ 

$\mathbb{C}_2 \; (= \delta(\mathbb{C}_1)) =$ 



$\mathbb{D}_2 \; (= \delta(\mathbb{D}_1)) =$ 

$\mathbb{E}_2 =$ 

$\mathbb{B}_1 =$ 

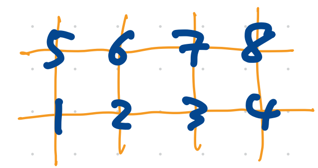
$$\mathbb{D}_2 = \mathcal{A}^{YY\to X}_{6\to 2}(\mathbb{C}_2) = \mathcal{A}^{YY\to X}_{4\to 3}(\mathbb{E}_2) \quad (1)$$

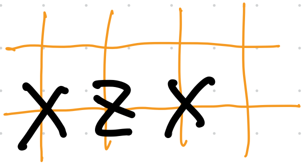
$$c_{\mathbb{D}_2} = -2i\{q_{\mathbb{C}_2} + q_{\mathbb{E}_2}\} \quad (2) \qquad q_{\mathbb{C}_2} + q_{\mathbb{E}_2} = 0 \quad (3)$$

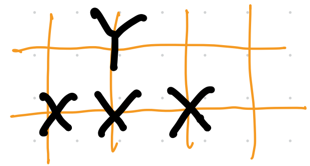
$$\mathbb{B}_1 = \mathcal{A}^{YY\to X}_{4\to 3}(\mathbb{C}_1) = \mathcal{A}^{YY\to X}_{1\to 2}(\mathbb{C}_2) \quad (4)$$

$$c_{\mathbb{B}_1} = -2i\{q_{\mathbb{C}_1} + q_{\mathbb{C}_2}\} \quad (5) \qquad q_{\mathbb{C}_1} + q_{\mathbb{C}_2} = 0 \quad (6)$$

$$
\left.\begin{aligned}
q_{C_1} + q_{E_2} &= 0 \quad (1)\\
q_{C_2} + q_{E_2} &= 0 \quad (2)\\
q_{C_1} + q_{C_2} &= 0 \quad (3)
\end{aligned}\right\} \Rightarrow \boxed{\underset{\hat{Y}\hat{Z}\hat{X}}{\underset{\shortparallel}{q_{C_1}}} = \underset{\hat{X}\hat{Z}\hat{Y}}{\underset{\shortparallel}{q_{C_2}}} = 0} \quad (4)
$$



$C_1' =$ 

$D_1' =$ 

$$D_1' = \mathcal{A}^{YY \to Z}_{6 \to 2}(C_1') \quad (5) \qquad \boxed{q_{C_1'} = 0} \quad (6)$$

$C_2' =$ 

$$\boxed{q_{C_2'} = q_{C_1'} = 0} \quad (7)$$

<general $k_{max} \geq 3$>

$$
\begin{array}{llll}
 & \text{odd } j & & \text{even } j \\
\hat{\boldsymbol{C}}_j & Y\,Z \text{-----------} Z\,Z\,X & \hat{\boldsymbol{C}}_j & X\,Z \text{-----------} Z\,Z\,Y \\
\hat{\boldsymbol{D}}_j & Y\,Z \text{----} Z\,\overset{Y}{X}\,Z \text{----} Z\,Z\,X & \hat{\boldsymbol{D}}_j & X\,Z \text{----} Z\,\overset{Y}{X}\,Z \text{----} Z\,Z\,Y \\
\hat{\boldsymbol{E}}_j & Y\,Z \text{----} Z\,\overset{Y}{X}\,Z \text{----} Z\,Y & \hat{\boldsymbol{E}}_j & X\,Z \text{----} Z\,\overset{Y}{X}\,Z \text{----} Z\,X
\end{array}
$$

$$
\text{only for odd } k \left\{
\begin{array}{ll}
\hat{\boldsymbol{D}}_{k-1} & \overset{Y}{Z}\,Z \text{-----------} Z\,Z\,Y \\
\hat{\boldsymbol{E}}_{k-1} & \overset{Y}{Z}\,Z \text{-----------} Z\,X
\end{array}
\right.
$$