The $S=\frac{1}{2}$ XY and XYZ models on the two or higher dimensional hypercubic lattice do not possess nontrivial local conserved quantities

Naoto Shiraishi and Hal Tasaki

webinar@YouTube, 2024

background

main results

about the proof

summary and dicussion

background

mathematical studies of quantum many-body systems

exact solutions free fermion, Bethe ansatz, Yang-Baxter relation ....

only cover special "integrable" models

rigorous, general theorems

cover models in a certain class, both integrable and non-integrable

there are properties/phenomena (quantum chaos, ETH = energy eigenstate thermalization hypothesis) that are expected to take place only in non-integrable systems

before 2019, there were essentially no mathematical results that exclusively applied to non-integrable systems

Shiraishi's work in 2019

$S=\frac{1}{2}$ **XYZ-h spin chain with Hamiltonian**

$$\hat{H}_{\text{XYZ-h}} = -\sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{X}_{j+1} + J_{\text{Y}}\,\hat{Y}_j\hat{Y}_{j+1} + J_{\text{Z}}\,\hat{Z}_j\hat{Z}_{j+1} + h\,\hat{Z}_j\}$$

integrable (can be mapped to a free fermion) if $J_{\text{Z}}=0$

serieses of local conserved quantities $[\hat{H}_{\text{XY-h}}, \hat{Q}^{\pm}_{k_{\text{max}}}]=0$

$k_{\text{max}}=3,4,\ldots$

$$\hat{Q}_3^+ = \sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{Z}_{j+1}\hat{X}_{j+2} + J_{\text{Y}}\,\hat{Y}_j\hat{Z}_{j+1}\hat{Y}_{j+2} - h(\hat{X}_j\hat{X}_{j+1}+\hat{Y}_j\hat{Y}_{j+1})\}$$

$$\hat{Q}_3^- = \sum_{j=1}^{L}\{\hat{X}_j\hat{Z}_{j+1}\hat{Y}_{j+2} - \hat{Y}_j\hat{Z}_{j+1}\hat{X}_{j+2}\}$$

$$\hat{Q}_4^+ = \sum_{j=1}^{L}\{J_{\text{X}}(\hat{X}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{X}_{j+3}+\hat{Y}_j\hat{Y}_{j+1}) + J_{\text{Y}}(\hat{Y}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{Y}_{j+3}+\hat{X}_j\hat{X}_{j+1}) - h(\hat{X}_j\hat{Z}_{j+1}\hat{X}_{j+2}+\hat{Y}_j\hat{Z}_{j+1}\hat{Y}_{j+2})\}$$

$$\hat{Q}_4^- = \sum_{j=1}^{L}\{\hat{X}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{Y}_{j+3} - \hat{Y}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{X}_{j+3}\}$$

if $J_{\text{X}}\neq J_{\text{Y}}$, $J_{\text{Z}}\neq 0$, and $h\neq 0$, the model has no local conserved quantities with support size $3\leq k_{\text{max}}\leq L/2$

Naoto Shiraishi, "Proof of the absence of local conserved quantities in the XYZ chain with a magnetic field", 2019

the first rigorous result that applies exclusively to non-integrable models!

Shiraishi's work and its extensions

$S=\frac{1}{2}$ XYZ-h spin chain with Hamiltonian

$$\hat{H}_{\text{XYZ-h}} = -\sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{X}_{j+1} + J_{\text{Y}}\,\hat{Y}_j\hat{Y}_{j+1} + J_{\text{Z}}\,\hat{Z}_j\hat{Z}_{j+1} + h\,\hat{Z}_j\}$$

if $J_{\text{X}} \neq J_{\text{Y}}$, $J_{\text{Z}} \neq 0$, and $h \neq 0$, the model has no local conserved quantities with support size $3 \leq k_{\text{max}} \leq L/2$

Shiraishi 2019

extensions to the quantum Ising model (Chiba 2024), the PXP model (Park and Lee 2024), the $S=1/2$ chains with next-nearest neighbor interactions (Shiraishi 2024), and the $S=1$model with bilinear biquadratic intereactions (Park and Lee 2024) ...

FAQ: do these results prove the models are non-integrable?

Answer: this may not be a good question. the answer depends on how one defines "integrability"

empirical rule: a simple quantum spin model is either integrable or does not possess local conserved quantities

Shiraishi's work and its extensions

$S = \frac{1}{2}$ … with Hamiltonian

$\hat{H}_{X}$ … $h\,\hat{Z}_j\}$

if … al

co… $\leq L/2$

…ishi 2019

exten…

model…

neigh…

biline…

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Shiraishi, 2024
Journal of Statistical Physics (open access!)

Check for updates

Absence of Local Conserved Quantity in the Heisenberg Model with Next-Nearest-Neighbor Interaction

Naoto Shiraishi[1]

Received: 1 April 2024 / Accepted: 13 August 2024

Abstract
We rigorously prove that the $S = 1/2$ anisotropic Heisenber… next-nearest-neighbor interaction, which is anticipated to be no… integrable in the sense that this system has no nontrivial loca… result covers some important models including the Majumdar–… Sutherland model, and many other zigzag spin chains as specia… shown to be non-integrable while they have some solvable energy… this result, we provide a pedagogical review of the proof of non-i… XYZ chain with Z magnetic field, whose proof technique is empl…

Keywords Integrable systems · Heisenberg chain · Majumdar–Gh… integral of motion

Shiraishi's work and its extensions

$S=\frac{1}{2}$ **XYZ-h spin chain with Hamiltonian**

$$\hat{H}_{\text{XYZ-h}} = -\sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{X}_{j+1} + J_{\text{Y}}\,\hat{Y}_j\hat{Y}_{j+1} + J_{\text{Z}}\,\hat{Z}_j\hat{Z}_{j+1} + h\,\hat{Z}_j\}$$

if $J_{\text{X}} \neq J_{\text{Y}}$, $J_{\text{Z}} \neq 0$, and $h \neq 0$, the model has no local conserved quantities with support size $3 \leq k_{\text{max}} \leq L/2$

Shiraishi 2019

extensions to the quantum Ising model (Chiba 2024), the PXP model ... the $S=1/2$ chains with next-nearest neighb ... with bilinea ...

FAQ: d ... ble?

we shall extend the proof of the absence of local conserved quantities to quantum spin models in two or higher dimensions

Answer: this may not be a good question. the answer depends on how one defines "integrability"

empirical rule: a simple quantum spin model is either integrable or does not possess local conserved quantities

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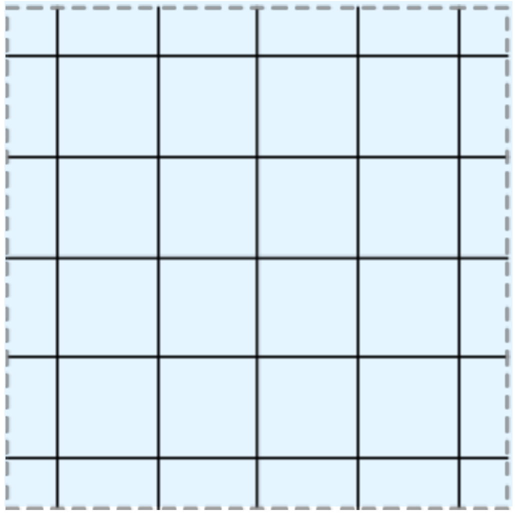
$S=\frac{1}{2}$ model in d dimensions

operators of a single spin is spanned by

$$\hat{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \hat{X}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \hat{Y}=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\quad \hat{Z}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$\Lambda=\{1,\ldots,L\}^d$ d-dimensional $L\times\ldots\times L$ hypercubic lattice with periodic b.c.



$\hat{X}_u,\hat{Y}_u,\hat{Z}_u$ copies of $\hat{X},\hat{Y},\hat{Z}$ at site $u\in\Lambda$

Hamiltonian of the XYZ model

$$\hat{H}=-\frac{1}{2}\sum_{\substack{u,v\in\Lambda\\(|u-v|=1)}}\{J_\mathrm{X}\,\hat{X}_u\hat{X}_v+J_\mathrm{Y}\,\hat{Y}_u\hat{Y}_v+J_\mathrm{Z}\,\hat{Z}_u\hat{Z}_v\}-\sum_{u\in\Lambda}\{h_\mathrm{X}\,\hat{X}_u+h_\mathrm{Y}\,\hat{Y}_u+h_\mathrm{Z}\,\hat{Z}_u\}$$

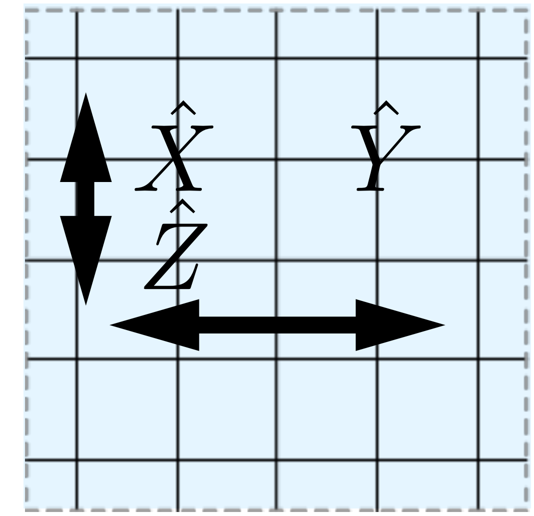
$$J_\mathrm{X},J_\mathrm{Y},J_\mathrm{Z},h_\mathrm{X},h_\mathrm{Y},h_\mathrm{Z}\in\mathbb{R},\qquad J_\mathrm{X}\neq 0,J_\mathrm{Y}\neq 0$$

local conserved quantities

$\boldsymbol{A} = \bigotimes_{u \in S} \hat{A}_u$ product of Pauli matrices

$\Lambda \supset S \neq \emptyset \qquad \hat{A}_u = \hat{X}_u, \hat{Y}_u, \hat{Z}_u$

$\boldsymbol{A} =$



$\mathrm{Wid}\boldsymbol{A} = 3$

$\mathcal{P}_\Lambda$ the set of all products

$\mathrm{Wid}\boldsymbol{A}$ the maximum width of the support $S \subset \Lambda$

candidate of a local conserved quantity with width k_{max} such that $2 \leq k_{\mathrm{max}} \leq \frac{L}{2}$

$$\hat{Q} = \sum_{\substack{\boldsymbol{A} \in \mathcal{P}_\Lambda \\ (\mathrm{Wid}\boldsymbol{A} \leq k_{\mathrm{max}})}} q_{\boldsymbol{A}} \, \boldsymbol{A} \qquad q_{\boldsymbol{A}} \in \mathbb{C}$$

$q_A \neq 0$ for at least one A with $\mathrm{Wid}A = k_{\mathrm{max}}$

$\hat{Q}$ is a local conserved quantity iff $[\hat{H}, \hat{Q}] = 0$

main theorems

$$\hat{H}=-\frac{1}{2}\sum_{\substack{u,v\in\Lambda\\(|u-v|=1)}}\{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v+J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v+J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\}-\sum_{u\in\Lambda}\{h_{\mathrm{X}}\,\hat{X}_u+h_{\mathrm{Y}}\,\hat{Y}_u+h_{\mathrm{Z}}\,\hat{Z}_u\}$$

$$\hat{Q}=\sum_{\substack{\boldsymbol{A}\in\mathcal{P}_\Lambda\\(\mathrm{Wid}\,\boldsymbol{A}\le k_{\max})}} q_{\boldsymbol{A}}\,\boldsymbol{A} \qquad q_{\boldsymbol{A}}\in\mathbb{C}$$

$\hat{Q}$ is a local conserved quantity iff $[\hat{H},\hat{Q}]=0$

Theorem: there are no local conserved quantities $\hat{Q}$ with width $k_{\max}$ such that $3\le k_{\max}\le\frac{L}{2}$

Hamiltonian is a local conserved quantity with $k_{\max}=2$

Theorem: any local conserved quantity with $k_{\max}=2$ is written as $\hat{Q}=\eta\,\hat{H}+\hat{Q}_1$ with $\eta\neq 0$, where $\hat{Q}_1$ is a linear combination of single-site Pauli matrices

See also the accompanying video!

background
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basic strategy of the proof Shiraishi 2019, 2024

$$\hat{H} = -\frac{1}{2}\sum_{\substack{u,v\in\Lambda \\ (|u-v|=1)}} \{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v + J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v + J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\Lambda}\{h_{\mathrm{X}}\,\hat{X}_u + h_{\mathrm{Y}}\,\hat{Y}_u + h_{\mathrm{Z}}\,\hat{Z}_u\}$$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$
$$\hat{X}\hat{Y} = -\hat{Y}\hat{X} = \mathrm{i}\hat{Z}$$
$$\hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = \mathrm{i}\hat{X}$$
$$\hat{Z}\hat{X} = -\hat{X}\hat{Z} = \mathrm{i}\hat{Y}$$

$$\boldsymbol{A} = \bigotimes_{u\in S}\hat{A}_u \qquad [\hat{H},\boldsymbol{A}] = \sum_{\boldsymbol{B}\in\mathcal{P}_\Lambda}\lambda_{\boldsymbol{A},\boldsymbol{B}}\,\boldsymbol{B}$$

written in terms of $J_{\mathrm{X}}, J_{\mathrm{Y}}, J_{\mathrm{Z}}, h_{\mathrm{X}}, h_{\mathrm{Y}}, h_{\mathrm{Z}}$

$$\hat{Q} = \sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda} q_{\boldsymbol{A}}\,\boldsymbol{A}$$

$$[\hat{H},\hat{Q}] = \sum_{\boldsymbol{B}\in\mathcal{P}_\Lambda}\Bigl(\sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda}\lambda_{\boldsymbol{A},\boldsymbol{B}}\,q_{\boldsymbol{A}}\Bigr)\boldsymbol{B}$$

$$[\hat{H},\hat{Q}] = 0$$

$\Updownarrow$

$\sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda}\lambda_{\boldsymbol{A},\boldsymbol{B}}\,q_{\boldsymbol{A}} = 0$ **for all** $\boldsymbol{B}\in\mathcal{P}_\Lambda$

coupled linear equations for q_A

$\Downarrow$

$q_{\boldsymbol{A}} = 0$ **for all** $\boldsymbol{A}\in\mathcal{P}_\Lambda$ when $3 \le k_{\max} \le \frac{L}{2}$

1st step of the proof Shiraishi 2019, 2024

$\sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda\,(\mathrm{Wid}\boldsymbol{A}\le \mathrm{k_{max}})} \lambda_{\boldsymbol{A},\boldsymbol{B}}\, q_{\boldsymbol{A}} = 0$ for all $\boldsymbol{B}\in\mathcal{P}_\Lambda$

if there is B such that $\lambda_{A,B}\neq 0$ for only one A

$$\lambda_{\boldsymbol{A},\boldsymbol{B}}\, q_{\boldsymbol{A}} = 0 \;\rightarrow\; q_{\boldsymbol{A}} = 0$$

if there is B such that $\lambda_{A,B}\neq 0$ only for $A = A_1, A_2$

$$\lambda_{\boldsymbol{A}_1,\boldsymbol{B}}\, q_{\boldsymbol{A}_1} + \lambda_{\boldsymbol{A}_2,\boldsymbol{B}}\, q_{\boldsymbol{A}_2} = 0 \;\rightarrow\; q_{\boldsymbol{A}_1} = \lambda\, q_{\boldsymbol{A}_2} \quad (\lambda\neq 0)$$

↓ Shiraishi-shift

lemma: for any A with $\mathrm{Wid}A = k_{\mathrm{max}}$, we have either

$$q_{\boldsymbol{A}} = 0,\;\; q_{\boldsymbol{A}} = \lambda' q_{\boldsymbol{C}_{\mathrm{XX}}},\;\; q_{\boldsymbol{A}} = \lambda'' q_{\boldsymbol{C}_{\mathrm{YX}}} \quad (\lambda', \lambda''\neq 0)$$

with

$$\boldsymbol{C}_{\mathrm{XX}} = \hat{X}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\mathrm{max}}-1)e_1}\hat{X}_{x_0+k_{\mathrm{max}}e_1}$$

$$\boldsymbol{C}_{\mathrm{YX}} = \hat{Y}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\mathrm{max}}-1)e_1}\hat{X}_{x_0+k_{\mathrm{max}}e_1} \quad e_1 = (1,0,\ldots,0)$$

the problem reduced to that in essentially one dimension!

2nd step of the proof

Shiraishi 2019, 2024

lemma: for any A with $\mathrm{Wid}A = k_{\max}$, we have either

$$q_{\boldsymbol{A}} = 0, \;\; q_{\boldsymbol{A}} = \lambda' q_{\boldsymbol{C}_{\mathrm{XX}}}, \;\; q_{\boldsymbol{A}} = \lambda'' q_{\boldsymbol{C}_{\mathrm{YX}}} \quad (\lambda', \lambda'' \neq 0)$$

with

$$\boldsymbol{C}_{\mathrm{XX}} = \hat{X}_{x_0+e_1} \hat{Z}_{x_0+2e_1} \cdots \hat{Z}_{x_0+(k_{\max}-1)e_1} \hat{X}_{x_0+k_{\max}e_1}$$

$$\boldsymbol{C}_{\mathrm{YX}} = \hat{Y}_{x_0+e_1} \hat{Z}_{x_0+2e_1} \cdots \hat{Z}_{x_0+(k_{\max}-1)e_1} \hat{X}_{x_0+k_{\max}e_1} \quad e_1 = (1, 0, \ldots, 0)$$

the problem reduced to that in essentially one dimension!

🎓 use $\sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda} \lambda_{\boldsymbol{A},\boldsymbol{B}}\, q_{\boldsymbol{A}} = 0$ for appropriate $\boldsymbol{B}$ to show

$$q_{\boldsymbol{C}_{\mathrm{XX}}} = q_{\boldsymbol{C}_{\mathrm{YX}}} = 0$$

$q_A = 0$ for any A with $\mathrm{Wid}A = k_{\max}$ ➡ contradiction

the step 🎓 requires us to solve a math puzzle

it is easier than the corresponding step in the one-dimensional problem, but we still need craftmanship...

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main results

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summary

☑ we proved that the XY and XYZ models on the d-dimensional hypercubic lattice with $d \geq 2$ possess no local conserved quantities

☑ the theorem applies to the simplest XX model

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\Lambda \\ (|u-v|=1)}} \{\hat{X}_u \hat{X}_v + \hat{Y}_u \hat{Y}_v\}$$

easily solved in 1D

quantum many-body models becomes "less solvable" in higher dimensions

☑ the strategy of the proof is a natural extension of that developed by Shiraishi in 2019, with some new ideas to treat higher dimensional models

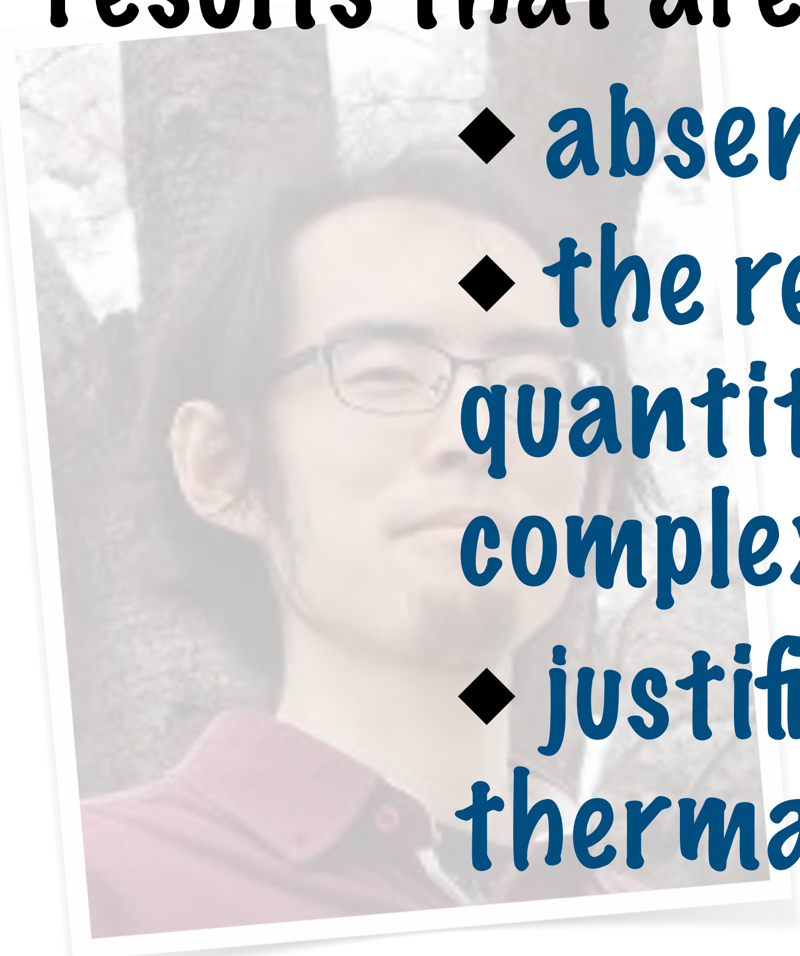
discussion

☑ proof of the absence of local conserved quantities is interesting and meaningful by itself

gives a strong indication that the models are not "solvable"

sheds light on the algebraic structure of quanum spin models

☑ it is challenging to develop techniques for proving results that are relevant to long-time dynamics



- absence of quasi local conserved quantities
- the relation between the lack of conserved quantities and the operator growth (Krylov complexity)
- justification of ETH (energy eigenstate thermalization hypothesis)