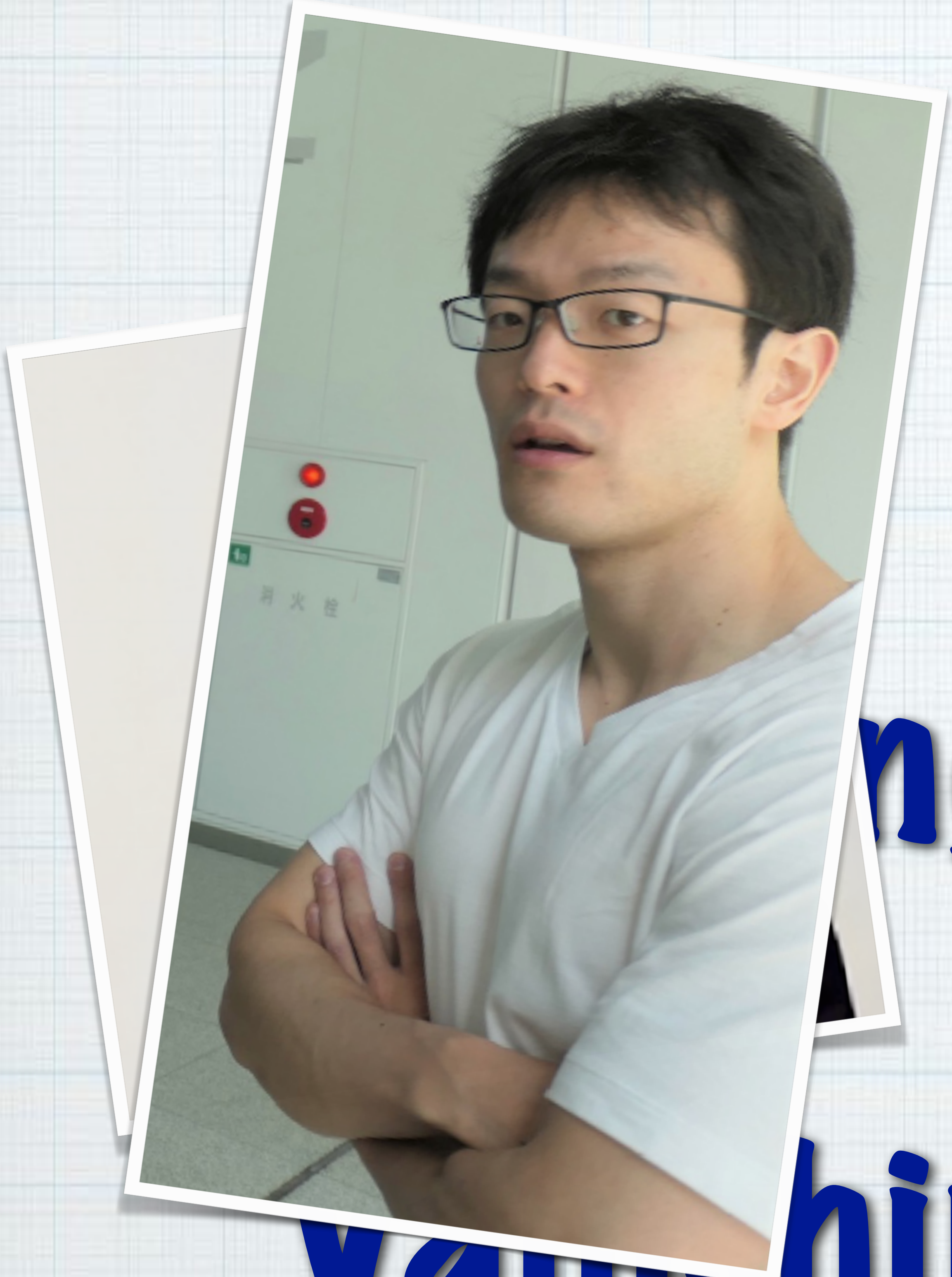


Off-Diagonal
ng Range Order
Implies
hing Charge Gap

Hal Tasaki and Haruki Watanabe

brief introduction @ YouTube / May 2021

previously known for lattice bosons with hard core interaction (Tian 1992)

- ☑ general inequality that relates the long range order (LRO) or the off-diagonal long range order (ODLRO) with the charge gap in a quantum many-body systems with U(1) symmetry
- ☑ applies to a very general class of systems including interacting particles (bosons or fermions) on lattice or in continuum, and various quantum spin systems
- ☑ different from the Goldstone theorem

two typical applications

interacting bosons

magnetization plateau

interacting bosons

bosons in a box (continuum or lattice) with volume V

$\hat{N}$ total particle number operator

$\hat{H}$ non-pathological Hamiltonian such that $[\hat{H}, \hat{N}] = 0$

ground state $|\mathrm{GS}_N\rangle$ g.s. energy E_N with N particles

nonzero charge gap means the system is insulating

charge gap

$$\Delta_N = E_{N+1} + E_{N-1} - 2E_N \simeq \frac{1}{V}\frac{\partial \mu}{\partial \rho}$$

$\Delta_N \geq \text{const} > 0 \longrightarrow |\mathrm{GS}_N\rangle$ has no ODLRO

$|\mathrm{GS}_N\rangle$ has ODLRO $\longrightarrow \Delta_N \leq \dfrac{\text{const}}{V} \Longleftrightarrow \dfrac{\partial \rho}{\partial \mu} \geq \text{const} > 0$

Bose-Einstein condensate is always "compressible"

interacting bosons

$|\mathrm{GS}_N\rangle$ has ODLRO $\Longrightarrow$ $\Delta_N \leq \dfrac{\mathrm{const}}{V}$ $\Longleftrightarrow$ $\dfrac{\partial \rho}{\partial \mu} \geq \mathrm{const} > 0$

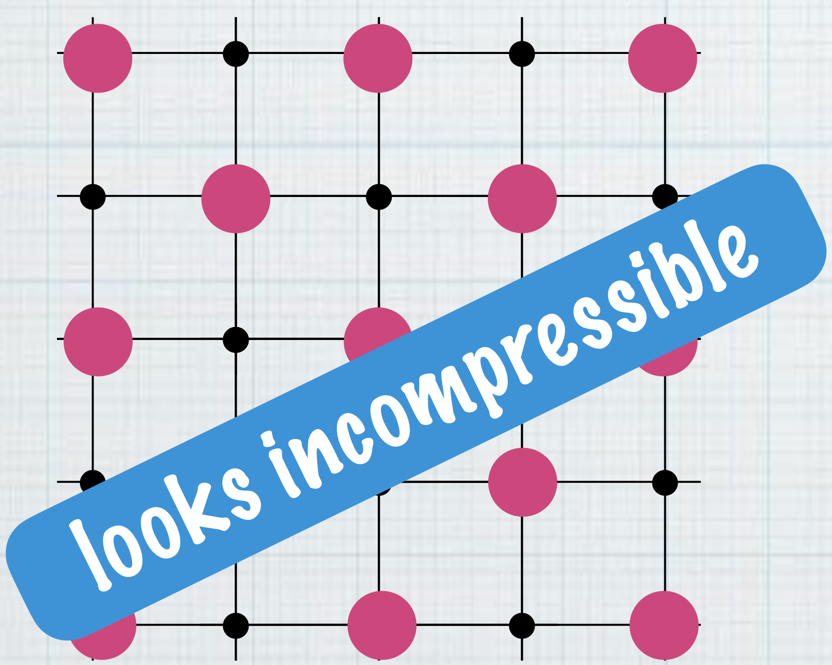
Bose-Einstein condensate is always "compressible"

commensurate supersolid

interesting application

example: soft-core Bose-Hubbard model with nearest-neighbor repulsion on the cubic lattice

$$N = \frac{(\text{number of sites})}{2}$$



$$\frac{\partial \rho}{\partial \mu} = 0$$

Mott insulator phase

broken translation symmetry

no BEC

$$\frac{\partial \rho}{\partial \mu} > 0$$

supersolid phase

broken translation symmetry

BEC

Ohgoe, Suzuki, Kawashima 2012

magnetization plateau

quantum spin system on a lattice with N sites

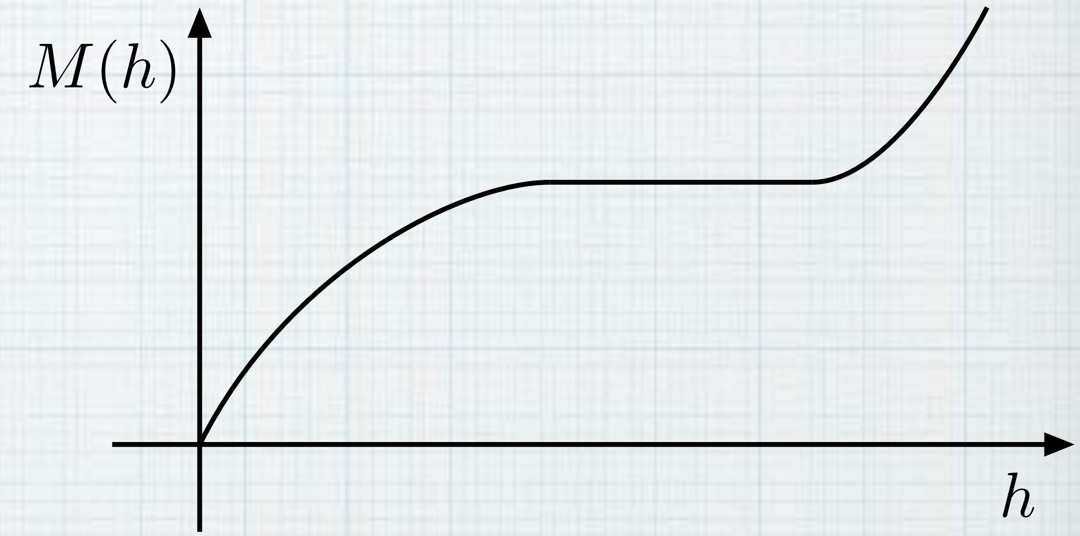
$\hat{H}_0$ non-pathological Hamiltonian such that $[\hat{H}_0, \hat{S}^{\mathrm{z}}_{\mathrm{tot}}] = 0$

$\hat{S}^{\mathrm{z}}_{\mathrm{tot}} = \sum_j \hat{S}^{\mathrm{z}}_j$

$|\mathrm{GS}_h\rangle$ ground state of

$$\hat{H}_h = \hat{H}_0 - h\,\hat{S}^{\mathrm{z}}_{\mathrm{tot}}$$

$$\hat{S}^{\mathrm{z}}_{\mathrm{tot}}|\mathrm{GS}_h\rangle = M(h)|\mathrm{GS}_h\rangle$$



when h is within a magnetization plateau

$$\sum_{j,k} \zeta_j \zeta_k \langle \mathrm{GS}_h|(\hat{S}^{\mathrm{x}}_j \hat{S}^{\mathrm{x}}_k + \hat{S}^{\mathrm{y}}_j \hat{S}^{\mathrm{y}}_k)|\mathrm{GS}_h\rangle \le \mathrm{const}\, N$$

for any $\zeta_j \in \mathbb{R}$ with $|\zeta_j| \le 1$

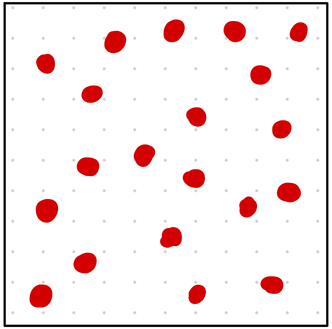
no long-range order in the xy directions

details

bosons in a box

basic setting

$\mathbb{r} \in [0, L]^3 \subset \mathbb{R}^3$ volume $V = L^3$



$\hat{\psi}(\mathbb{r})$ annihilation
$\hat{\psi}^\dagger(\mathbb{r})$ creation
operator of a boson at $\mathbb{r}$

$$[\hat{\psi}(\mathbb{r}), \hat{\psi}(\mathbb{s})] = 0, \quad [\hat{\psi}(\mathbb{r}), \hat{\psi}^\dagger(\mathbb{s})] = \delta(\mathbb{r} - \mathbb{s})$$

$\mathbb{r}, \mathbb{s} \in [0, L]^3$

number operator

$$\hat{N} = \int d^3\mathbb{r}\, \hat{\psi}^\dagger(\mathbb{r}) \hat{\psi}(\mathbb{r})$$

Hamiltonian

$$\hat{H} = \int d^3\mathbb{r}\, \hat{\psi}^\dagger(\mathbb{r}) \left\{ -\frac{\triangle}{2m} + U(\mathbb{r}) \right\} \hat{\psi}(\mathbb{r}) + \frac{1}{2} \int d^3\mathbb{r}\, d^3\mathbb{s}\, \hat{\psi}^\dagger(\mathbb{r}) \hat{\psi}^\dagger(\mathbb{s}) V_{int}(\mathbb{r} - \mathbb{s}) \hat{\psi}(\mathbb{s}) \hat{\psi}(\mathbb{r})$$

$U(\mathbb{r})$ single particle potential $\quad V_{int}(\mathbb{r})$ interaction potential

$$[\hat{H}, \hat{N}] = 0$$

(iii) the ground state and the charge gap

$[\hat{H}, \hat{N}] = 0$

$|GS_N\rangle$ a ground state
E_N the ground state energy
} within the N particle sector

chemical potential $\mu_N = E_{N+1} - E_N$

charge gap $\Delta_N = \mu_N - \mu_{N-1} = E_{N+1} + E_{N-1} - 2E_N$

($\Delta_N = O(1) \rightarrow$ the ground state is insulating)

(iv) off-diagonal long range order (ODLRO)

order operator $\hat{O} = \int d^3r\, \hat{\psi}(r)$

(can be generalized to $\hat{O} = \int d^3r\, \xi(r) \hat{\psi}(r)$)

$[\hat{O}, \hat{O}^\dagger] = V \qquad [\hat{N}, \hat{O}^\dagger] = \hat{O}^\dagger$

$$\boxed{\langle GS_N | \hat{O}^\dagger \hat{O} | GS_N \rangle = \begin{cases} O(V) & \text{NO ODLRO} \\ O(V^2) & \text{ODLRO} \end{cases}}$$

NO ODLRO → normal

ODLRO → Bose-Einstein condensate

(ii) proof of the inequality

$$\hat{O}^\dagger = \int d^3r\, \hat{\psi}^\dagger(r)$$

$\hat{O}^\dagger |GS_N\rangle$ is a $(N+1)$-particle state

variational principle $\Rightarrow$
$$\frac{\langle GS_N | \hat{O} \hat{H} \hat{O}^\dagger | GS_N\rangle}{\langle GS_N | \hat{O} \hat{O}^\dagger | GS_N\rangle} \geq E_{N+1}$$

$$\langle GS_N | \hat{O} \{\hat{H} - E_N\} \hat{O}^\dagger | GS\rangle_N \geq \overbrace{(E_{N+1} - E_N)}^{=\mu_N} \overbrace{\langle GS_N | \hat{O} \hat{O}^\dagger | GS_N\rangle}^{= V + \langle GS_N | \hat{O}^\dagger \hat{O} | GS_N\rangle} \quad ☆$$

similarly

$$\langle GS_N | \hat{O}^\dagger \{\hat{H} - E_N\} \hat{O} | GS\rangle_N \geq \overbrace{(E_{N-1} - E_N)}^{= -\mu_{N-1}} \langle GS_N | \hat{O}^\dagger \hat{O} | GS_N\rangle \quad ☆☆$$

☆ + ☆☆

$$\underbrace{\langle GS_N | [\hat{O}, [\hat{H}, \hat{O}^\dagger]] | GS_N\rangle}_{\leq AV + BN} \geq \overbrace{\boxed{(\mu_N - \mu_{N-1})}}^{= \Delta_N} \langle GS_N | \hat{O}^\dagger \hat{O} | GS_N\rangle + \mu_N V$$

with $A = \frac{1}{V} \int d^3r\, |U(r)|$ $\quad B = 2 \int d^3r\, |V_{int}(r)|$

$$\langle GS_N | [\hat{O}, [\hat{H}, \hat{O}^\dagger]] | GS_N \rangle \geq \underbrace{(\mu_N - \mu_{N-1})}_{= \Delta_N} \langle GS_N | \hat{O}^\dagger \hat{O} | GS_N \rangle + \mu_N V$$

$$\leq AV + BN = (A + B\rho) V \qquad \rho = N/V$$

$$\boxed{\Delta_N \langle GS_N | \hat{O}^\dagger \hat{O} | GS_N \rangle \leq (A + B\rho + |\mu_N|) V}$$

main inequality

$$C := A + B\rho + |\mu_N| = O(1)$$

(i) if Δ_N is positive and $O(1)$

$$\langle GS_N | \hat{O}^\dagger \hat{O} | GS_N \rangle \leq \frac{C}{\Delta_N} V = O(V) \Longrightarrow \text{NO ODLRO}$$

(ii) if $^\exists$ODLRO, i.e., $\langle GS_N | \hat{O}^\dagger \hat{O} | GS_N \rangle \geq \eta V^2$ with $\eta \gneq 0$

$$\Delta_N \leq \frac{C}{\eta} \frac{1}{V} \Longrightarrow \text{vanishing charge gap}$$

$$\frac{\partial \rho}{\partial \mu} \leftarrow \chi_N = \frac{1}{V} \frac{1}{\mu_N - \mu_{N-1}} = \frac{1}{V} \frac{1}{\Delta_N} \geq \frac{\eta}{C} \Rightarrow \text{"compressible"}$$

summary

- inequality that relates the long range order (LRO) or the off-diagonal long range order (ODLRO) with the charge gap in a large class of quantum many-body systems with U(1) symmetry

previously known for lattice bosons with hard core interaction (Tian 1992)

- bosons or fermions on lattice or in continuum
  - if the charge gap is nonzero there is no ODLRO
  - if there is ODLRO the charge gap is vanishing and the ground state is "compressible"
- quantum spin systems
  - a ground state within a magnetization plateau does not have transverse LRO