Integrable and non-integrable quantum spin chains

part II proof of the absence of nontrivial local conserved quantities in the model with $h \neq 0$

Advanced Topics in Statistical Physics

by Hal Tasaki

trivial local conserved quantities = Hamiltonian, magnetization

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§ notations and main theorem

$S=\frac{1}{2}$ quantum spin system on $\Lambda=\{1,2,\cdots,L\}$

$$\hat{H}=\sum_{u=1}^{L}\left(\hat{X}_u\hat{X}_{u+1}+\hat{Y}_u\hat{Y}_{u+1}+h\hat{X}_u\right) \quad (1) \qquad \hat{X}_{L+1}=\hat{X}_1,\ \hat{Y}_{L+1}=\hat{Y}_1$$

products (of Pauli operators)

$$\hat{A}=\bigotimes_{u\in\mathrm{supp}\hat{A}}\hat{A}_u \quad (2) \qquad \mathrm{supp}\hat{A}\subset\Lambda,\ \hat{A}_u=\hat{X}_u,\hat{Y}_u,\hat{Z}_u$$

$\mathcal{P}$: the set of all products (note that $\mathcal{P}\ni\hat{I}$)

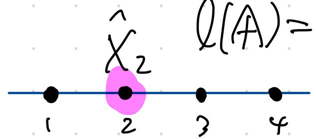
$\mathcal{P}$ is a basis of the whole space of operators on Λ

$\ell(\hat{A})$ length of $\hat{A}$

periodic b.c.

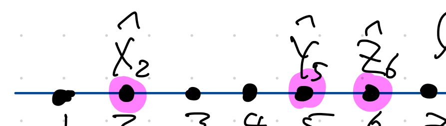
. minimum ℓ s.t. $\mathrm{supp}\hat{A}\subset\{a,\cdots,a+\ell-1\}$ for some a

$\hat{A}=\hat{X}_2$



$\ell(\hat{A})=1$

$\hat{B}=\hat{X}_2\hat{Y}_5\hat{Z}_6$



$\ell(\hat{B})=5$

local conserved quantity with (maximum) length k

$$\hat{Q} = \sum_{\substack{\hat{A} \in \mathcal{P} \\ (\ell(A) \leq k)}} q_A \hat{A} \quad (1) \qquad q_A \in \mathbb{C}, \ q_A \neq 0 \text{ for some } \hat{A} \text{ with } \ell(\hat{A}) = k$$

(see 1b-p.9)

$$\hat{Q} \text{ is a local conserved quantity} \iff [\hat{H}, \hat{Q}] = 0 \quad (2)$$

Theorem (Yamaguchi, Chiba, Shiraishi, 2024) if $h \neq 0$, there are no local consered quantities with maximum length k s.t. $3 \leq k \leq \frac{L}{2}$

- strongly suggests that the model with $h \neq 0$ is non-integrable
- $\hat{H}$ is a local conserved quantity with $k = 2$

  one may also prove that the only local conserved quantity with $1 \leq k \leq \frac{L}{2}$ is $q\hat{H}$ constant

We prove the theorem as in the original paper extension of Shiraishi 2019 for the XYZ-h model

proof is elementary!

§ basic relations and a useful lemma

commutators with pieces of Hamiltonian

$$\hat{h} = \hat{X}_u \hat{X}_{u+1},\ \hat{Y}_u \hat{Y}_{u+1},\ \hat{X}_u \quad \in P$$

$$\hat{A} \in P \qquad [\hat{h}, \hat{A}] = \begin{cases} 0 \\ (\text{nonzero const.})\ \hat{B} \in P \end{cases}$$

" $\hat{A}$ generates $\hat{B}$ (with $\hat{h}$) "

examples $\hat{A} = \hat{X}_2 \hat{Y}_3 \hat{Z}_4 \hat{Y}_5$ $\quad \ell(\hat{A}) = 4$

supp $\hat{B} \supsetneq$ supp $\hat{A}$

i) $[\hat{Y}_1 \hat{Y}_2, \hat{A}] = -2i\, \hat{Y}_1 \hat{Z}_2 \hat{Y}_3 \hat{Z}_4 \hat{Y}_5$, $\quad [\hat{X}_5 \hat{X}_6, \hat{A}] = 2i\, \hat{X}_2 \hat{Y}_3 \hat{Z}_4 \hat{Z}_5 \hat{X}_6$

supp $\hat{B}$ = supp $\hat{A}$

ii) $[\hat{X}_3, \hat{A}] = 2i\, \hat{X}_2 \hat{Z}_3 \hat{Z}_4 \hat{Y}_5$, $\quad [\hat{X}_4, \hat{A}] = -2i\, \hat{X}_2 \hat{Y}_3 \hat{Y}_4 \hat{Y}_5$

supp $\hat{B} \subsetneq$ supp $\hat{A}$

iii) $[\hat{Y}_3 \hat{Y}_4, \hat{A}] = 2i\, \hat{X}_2 \hat{X}_4 \hat{Y}_5$, $\quad [\hat{X}_2 \hat{X}_3, \hat{A}] = 2i\, \hat{Z}_3 \hat{Z}_4 \hat{Y}_5$

$$[\hat{Y}_3 \hat{Y}_4, \hat{Y}_3 \hat{Z}_4] = (\hat{Y}_3)^2 [\hat{Y}_4, \hat{Z}_4]$$

note

- $[\hat{X}_3 \hat{X}_4, \hat{A}] = 0 \quad \because \hat{X}_3 \hat{X}_4 \hat{Y}_3 \hat{Z}_4 = \hat{X}_3 \hat{Y}_3 \hat{X}_4 \hat{Z}_4 = (-\hat{Y}_3 \hat{X}_3)(-\hat{Z}_4 \hat{X}_4) = \hat{Y}_3 \hat{Z}_4 \hat{X}_3 \hat{X}_4$

basic relations

$$\hat{A} \in P, \quad [\hat{H}, \hat{A}] = \sum_{\hat{B} \in P} \boxed{\lambda_{A,B}} \, \hat{B} \qquad (1)$$

$\lambda_{A,B}$ → nonzero if $\hat{A}$ generates $\hat{B}$

e.g.

$$[\hat{H}, \hat{X}_2 \hat{Y}_3] = -2i \hat{Y}_1 \hat{Z}_2 \hat{Y}_3 + 2i \hat{X}_2 \hat{Z}_3 \hat{X}_4 + 2i \hat{Z}_3 - 2i \hat{Z}_2 + 2i h \hat{X}_2 \hat{Z}_3 \qquad (2)$$

$$\hat{Q} = \sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \leq k)}} q_A \hat{A} \qquad (3)$$

$$[\hat{H}, \hat{Q}] = \sum_{\hat{A}} q_A [\hat{H}, \hat{A}] = \sum_{\hat{A}} q_A \sum_{\hat{B}} \lambda_{A,B} \hat{B} = \sum_{\hat{B}} \left(\sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \leq k)}} \lambda_{A,B} q_A \right) \hat{B} \qquad (4)$$

we find

$$[\hat{H}, \hat{Q}] = 0 \iff \sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \leq k)}} \lambda_{A,B} q_A = 0 \quad (\star) \quad \text{for all } \hat{B} \in P \qquad (5)$$

$\hat{Q}$ is a conserved quantity

basic relation

coupled linear equations for q_A

lemma 1 if there are $\hat{A}, \hat{B} \in P$ s.t. $\hat{A}$ is the only product with length $\leq k$ that generates $\hat{B}$, then $q_A = 0$

proof $(\star)$ for $\hat{B}$: $\lambda_{A,B} q_A = 0$ thus $q_A = 0$ //

§ Strategy of the proof (Shiraishi 2019)

- fix k s.t. $3 \le k \le \frac{L}{2}$
- assume there is a local conserved quantity with maximum length k

$$\hat{Q} = \sum_{A\,(\ell(A) \le k)} q_A \hat{A} \quad (q_A \neq 0 \text{ for some } \hat{A} \text{ with } \ell(A) = k)$$

use selected sets of basic relations ☆ → $\displaystyle\sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \le k)}} \lambda_{A,B}\, q_A = 0$ p. 4

step 1 use Shiraishi shift to show $q_A = 0$ for any A with $\ell(A) = k$ unless $\hat{A}$ is of a "standard form".

step 2 show $q_A = 0$ for any $\hat{A}$ with $\ell(\hat{A}) = k$ of a standard form

$\Downarrow$

$q_A = 0$ for any $\hat{A}$ with $\ell(\hat{A}) = k$ → contradiction!

There exist no local conserved quantity with maximum length k

§ step1 : Shiraishi shift

$\hat{A} \in P$ with $\ell(\hat{A}) = k$ $\mathrm{supp}\,\hat{A} \subset \{\underline{u}, \dots, \bar{u}\}$ with $\bar{u} = \underline{u} + k - 1$

left-most   right-most

the maxicum length in $\hat{Q}$

($\underline{u}, \bar{u}$ are unique since $k \leq \frac{L}{2}$)

lemma 2 $\;\; q_A = 0$ unless both (1) and (2) hold

(1) $\hat{A}_{\underline{u}} = \hat{X}_{\underline{u}}$ or $\hat{Y}_{\underline{u}}$, $\quad \hat{A}_{\underline{u}+1} \neq \hat{I}_{\underline{u}+1}$, $\hat{A}_{\underline{u}+1} \neq \hat{A}_{\underline{u}}$

(2) $\hat{A}_{\bar{u}} = \hat{X}_{\bar{u}}$ or $\hat{Y}_{\bar{u}}$, $\quad \hat{A}_{\bar{u}-1} \neq \hat{I}_{\bar{u}-1}$, $\hat{A}_{\bar{u}-1} \neq \hat{A}_{\bar{u}}$

proof let

$$\hat{B} = \begin{cases} -\frac{1}{2i}[\hat{Y}_{\bar{u}}\hat{Y}_{\bar{u}+1}, A] & \text{if } \hat{A}_{\bar{u}} = \hat{X}_{\bar{u}} \\ \frac{1}{2i}[\hat{X}_{\bar{u}}\hat{X}_{\bar{u}+1}, A] & \text{if } \hat{A}_{\bar{u}} = \hat{Y}_{\bar{u}} \\ -\frac{1}{2i}[\hat{X}_{\bar{u}}\hat{X}_{\bar{u}+1}, A] & \text{if } \hat{A}_{\bar{u}} = \hat{Z}_{\bar{u}} \end{cases}$$

- $\ell(\hat{B}) = k+1$
- $\hat{A}$ generates $\hat{B}$

$k=4$

	$\underline{u}$			$\bar{u}$	$\bar{u}+1$
$\hat{A}$	X	Z	Y	X	
$\hat{B}$	X	Z	Y	Z	Y
$\hat{A}'$		Y	Y	Z	Y

is there $\hat{A}'$ with $\ell(\hat{A}') \leq k$ that generates $\hat{B}$? $\rightarrow$ if not $q_A = 0$ (from lemma 1)

$\hookrightarrow$ only possible when $\mathrm{supp}\,\hat{A}' \subset \{\underline{u}+1, \dots, \bar{u}+1\}$ and $\;\; \ell(\hat{A}') = k$

$[\hat{X}_{\underline{u}}\hat{X}_{\underline{u}+1}, \hat{A}'] = \pm 2i\,\hat{B}$ or $[\hat{Y}_{\underline{u}}\hat{Y}_{\underline{u}+1}, \hat{A}'] = \pm 2i\,\hat{B}$ $\rightarrow$ (1) is satisfied

switch left $\leftrightarrow$ right $\Rightarrow$ (2) is necessary for $q_A \neq 0$ //

Shiraishi shift $\quad \hat{A} \in P$ with $\ell(\hat{A}) = k$ satisfies (1) and (2) of lemma 2

$$\hat{B} = \begin{cases} -\frac{1}{2i}[\hat{Y}_{\bar{u}} \hat{Y}_{\bar{u}+1}, A] & \text{if } \hat{A}_{\bar{u}} = \hat{X}_{\bar{u}} \\ \frac{1}{2i}[\hat{X}_{\bar{u}} \hat{X}_{\bar{u}+1}, A] & \text{if } \hat{A}_{\bar{u}} = \hat{Y}_{\bar{u}} \end{cases} \quad (1)$$

$\ell(B) = k+1$

A	X Z Y Y Z X Y
B	X Z Y Y Z X Z X
A'	Y Y Y Z X Z X

$\hat{A}'$ unique product that generates $\hat{B} \in P$ as

$$\hat{B} = \begin{cases} \pm \frac{1}{2i}[\hat{X}_{\underline{u}} \hat{X}_{\underline{u}+1}, \hat{A}'] \\ \pm \frac{1}{2i}[\hat{Y}_{\underline{u}} \hat{Y}_{\underline{u}+1}, \hat{A}'] \end{cases} \quad (2)$$

$\ell(A') = k$

$\mathcal{S}(\hat{A}) = A'$ Shiraishi shift of $\hat{A}$

basic relation ☆ for $\hat{B}$ $\rightarrow$ $\pm 2i\, q_A \pm 2i\, q_{A'} = 0$ (3) $\rightarrow$ $q_A = \pm q_{A'}$ (4)

✎ if $\hat{A}$ does not satisfy (1) or (2) of lemma 2, we say $\mathcal{S}(\hat{A})$ does not exist

$\rightarrow q_A = 0$

lemma 3 $\quad \hat{A} \in P$ with $\ell(\hat{A}) = k$

if $\mathcal{S}(\hat{A})$ does not exist then $q_A = 0$

if $\mathcal{S}(\hat{A})$ exists then $q_A = \pm q_{\mathcal{S}(A)}$

☆

$$\sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \le k)}} \lambda_{A,B}\, q_A = 0$$

Shiraishi shift — examples $h=5$

$\hat{A}$ X Z Z Z Y
X Z Z Z Z X
$\delta(\hat{A})$ Y Z Z Z Y
Y Z Z Z Z Y
$\delta^2(\hat{A})$ X Z Z Z Y

↓

can be continued forever

$\hat{A}$ X Y Y Z X
X Y Y Z Z Y
$\delta(\hat{A})$ Z Y Z Z Y
Z Y Z Z Z X

$\hat{A}$ X Z Z X
X Z Z Z Y
$\delta(\hat{A})$ Y Z Z Y
Y Z Z Z X

$\delta(\hat{A})$ does not satisfy (2) of lemma 2.

$q_{\delta(A)} = 0$ lemma 2

lemma 3

$q_A = \pm q_{\delta(A)} = 0$

a necessary condition that $\delta(\hat{A})$ can be shifted

(2)' $\hat{A}_{\underline{u}} = \hat{X}_{\underline{u}}$ or $\hat{Y}_{\underline{u}}$, $\hat{A}_{\underline{u}+1} = \hat{Z}_{\underline{u}+1}$, $(\hat{A}_{\underline{u}+2} \neq \hat{I}_{\underline{u}+2})$

a necessary and sufficient condition that $\hat{A}$ can be shifted indefinitely

$\hat{A}_{\underline{u}} = \hat{X}_{\underline{u}}$ or $\hat{Y}_{\underline{u}}$, $\hat{A}_{\bar{u}} = \hat{X}_{\bar{u}}$ or $\hat{Y}_{\bar{u}}$, $\hat{A}_u = \hat{Z}_u$ for $\underline{u} < u < \bar{u}$

lemma 4 $\hat{A} \in P$ with $\ell(\hat{A}) = k$

$q_A = 0$ unless $\hat{A} = \begin{cases} \hat{X}_{\underline{u}} \otimes \left(\bigotimes_{u=\underline{u}+1}^{\bar{u}-1} \hat{Z}_u \right) \otimes \hat{X}_{\bar{u}} & XX \\ \hat{X}_{\underline{u}} \otimes \left(\bigotimes_{u=\underline{u}+1}^{\bar{u}-1} \hat{Z}_u \right) \otimes \hat{Y}_{\bar{u}} & XY \\ \hat{Y}_{\underline{u}} \otimes \left(\bigotimes_{u=\underline{u}+1}^{\bar{u}-1} \hat{Z}_u \right) \otimes \hat{X}_{\bar{u}} & YX \\ \hat{Y}_{\underline{u}} \otimes \left(\bigotimes_{u=\underline{u}+1}^{\bar{u}-1} \hat{Z}_u \right) \otimes \hat{Y}_{\bar{u}} & YY \end{cases}$

standard forms

$$\left(\hat{Q} = \sum_{\substack{\hat{A} \in P \\ (\ell(A) \le k)}} q_A \hat{A}, \quad [\hat{H}, \hat{Q}] = 0 \right)$$

remark all the results apply to the integrable model with $h=0$

this consideration completely determines possible local conserved quantities!

a conserved quantity with $k=3$

$$\hat{Q}_3 = \sum_{u=1}^{L} \left(\hat{X}_u \hat{Z}_{u+1} \hat{X}_{u+2} + \hat{Y}_u \hat{Z}_{u+1} \hat{Y}_{u+2} \right)$$

part 1b - p9 - (9)

§ step 2 redefine coordinate and set $u=1$

$$\mathbb{C}_1 = \hat{Y}_1 \hat{Z}_2 \cdots \hat{Z}_{k-1} \hat{Y}_k \quad (1)$$

YY

XX

$$\mathbb{C}_j = \delta^{j-1}(\mathbb{C}_1) = \begin{cases} \hat{X}_j \hat{Z}_{j+1} \cdots \hat{Z}_{k+j-2} \hat{X}_{k+j-1} & (j \text{ even}) \\ \hat{Y}_j \hat{Z}_{j+1} \cdots \hat{Z}_{k+j-2} \hat{Y}_{k+j-1} & (j \text{ odd}) \end{cases} \quad (2)$$

relations between $q_{\mathbb{C}_j}$

even j

with $\hat{Y}_{k+j-1} \hat{Y}_{k+j}$

$$\mathbb{B} \quad \begin{array}{ccccc} j & & & & k+j \\ X & Z & \cdots\cdots & Z & Y \end{array} \quad (\ell(\mathbb{B}) = k+1)$$

with $\hat{X}_j \hat{X}_{j+1}$

$$\mathbb{C}_j \quad \begin{array}{cccc} X & Z \cdots & Z & X \\ j & & & k+j-1 \end{array} \qquad (\ell(\mathbb{C}_j) = \ell(\mathbb{C}_{j+1}) = k) \qquad \mathbb{C}_{j+1} \quad \begin{array}{cccc} Y & Z \cdots & Z & Y \\ j+1 & & & k+j \end{array}$$

basic relation ⊛ for $\mathbb{B}$ $\Rightarrow$ $-2i\, q_{\mathbb{C}_j} + 2i\, q_{\mathbb{C}_{j+1}} = 0$ (3)

$$q_{\mathbb{C}_{j+1}} = q_{\mathbb{C}_j} \quad (4)$$

⊛ $\displaystyle\sum_{\substack{\hat{A} \in P \\ (\ell(\hat{A}) \le k)}} \lambda_{A,B}\, q_A = 0$

odd j

we similarly get $q_{\mathbb{C}_{j+1}} = q_{\mathbb{C}_j}$ (5)

$\boxed{q_{\mathbb{C}_j} = q}$ (6) for all j

generation by the magnetic field term $h\hat{X}_u$

we treat the case with odd k $(3 \le k \le \frac{L}{2})$

$\mathbb{D}_j = -\frac{1}{2i}[\hat{X}_{k-1}, \hat{\mathbb{C}}_j] \in \mathcal{P}$ (1) for $j = 1, \dots, k-2$

$\ell(\mathbb{D}_j) = k$ (2) $\quad \mathbb{C}_j$ generates $\mathbb{D}_j$ $\quad \lambda_{\mathbb{C}_j,\mathbb{D}_j} = -2ih$ (3)

products with length $\le k$ that generate $\mathbb{D}_j$

$(\ell(\mathbb{C}_j) = \ell(\mathbb{D}_j) = k)$

even j

$$\begin{array}{lccccccccccccc} & j & & & & & k-1 & & & & & k+j-1 \\ \mathbb{C}_j & X & Z & Z & \cdots & Z & Z & Z & \cdots & Z & Z & X \\ \mathbb{D}_j & X & Z & Z & \cdots & Z & Y & Z & \cdots & Z & Z & X \end{array}$$

with $h\hat{X}_{k-1}$

$\mathbb{E}_j$ $\quad\quad \mathbb{E}_{j+1}$

$$\begin{array}{ccccccccccc} j & & & & & & & & & & k+j-2 \\ X & Z & Z & \cdots & Z & Y & Z & \cdots & Z & Y \end{array} \qquad \begin{array}{ccccccccccc} j+1 & & & & & & & & & & k+j-1 \\ Y & Z & \cdots & Z & Y & Z & \cdots & Z & Z & X \end{array}$$

with $\hat{X}_{k+j-2}\hat{X}_{k+j-1}$ $\quad$ with $\hat{X}_j\hat{X}_{j+1}$

$(\ell(\mathbb{E}_j) = k-1)$ $\quad (\ell(\mathbb{E}_{j+1}) = k-1)$

many other $\hat{A}$ with $\ell(\hat{A}) = k$ NOT in the standard forms generate $\mathbb{D}_j$ but they all have $q_A = 0$ because of lemma 4.5!

basic relation ⊗ for $\mathbb{D}_j$ $\Longrightarrow$ $-2ih\,q_{\mathbb{C}_j} + 2i\,q_{\mathbb{E}_j} + 2i\,q_{\mathbb{E}_{j+1}} = 0$ (4)

the "short product" $\mathbb{E}_j$ $\quad \ell(\mathbb{E}_j)=k-1$

$$\mathbb{E}_j = -\frac{1}{2i}\left[\hat{X}_{k-1}, \delta^{j-1}(\hat{Y}_1\hat{Z}_2\cdots\hat{Z}_{k-2}\hat{X}_{k-1})\right]\in P \quad (1) \qquad j=2,\cdots,k-2$$

the case $k=5$

	1 2 3 4 5		2 3 4 5 6		3 4 5 6 7
$\mathbb{C}_1$	Y Z Z Z Y	$\mathbb{C}_2$	X Z Z Z X	$\mathbb{C}_3$	Y Z Z Z Y
$\mathbb{D}_1$	Y Z Z Y Y	$\mathbb{D}_2$	X Z Y Z X	$\mathbb{D}_3$	Y Y Z Z Y
		$\mathbb{E}_2$	X Z Y Y	$\mathbb{E}_3$	Y Y Z X

products with length $\leq k$ that generate $\mathbb{D}_j$

odd j

$j \neq 1, k-2$

$\mathbb{C}_j$ $\quad \overset{j}{Y} Z Z \cdots Z \overset{k-1}{Z} Z \cdots Z Z \overset{k+j-1}{Y}$ $\quad$ with $\hbar \hat{X}_{k-1}$

$\mathbb{D}_j$ $\quad Y Z Z \cdots Z Y Z \cdots Z Z Y$

$\mathbb{E}_j$ $\quad Y Z Z \cdots Z Y Z \cdots Z X$ $\quad$ with $\hat{Y}_{k+j-2}\hat{Y}_{k+j-1}$

$\mathbb{E}_{j+1}$ $\quad X Z \cdots Z Y Z \cdots Z Z Y$ $\quad$ with $\hat{Y}_j\hat{Y}_{j+1}$

basic relation ⊛ for $\mathbb{D}_j$ $\Longrightarrow$ $-2i\hbar\, q_{\mathbb{C}_j} - 2i\, q_{\mathbb{E}_j} - 2i\, q_{\mathbb{E}_{j+1}} = 0$ $\quad (2)$

products with length $\leq k$ that generate $\mathbb{D}_j$

$j=1$

$$\mathbb{C}_1 \quad \overset{1}{Y} Z Z \cdots Z \overset{k-1}{Z} \overset{k}{Y}$$

with $\hbar \hat{X}_{k-1}$

$$\mathbb{D}_1 \quad Y Z Z \cdots Z Y Y$$

with $\hat{Y}_1 \hat{Y}_2$

$$\overset{2}{X} Z \cdots Z Y \overset{k}{Y} \quad \mathbb{E}_2$$

basic relation ⊛ for $\mathbb{D}_1$ $\Rightarrow$ $-2i\hbar\, q_{\mathbb{C}_1} - 2i\, q_{\mathbb{E}_2} = 0$ (1)

$j=k-2$

$$\mathbb{C}_{k-2} \quad \overset{k-2}{Y} \overset{k-1}{Z} Z \cdots Z Z \overset{2k-3}{Y}$$

with $\hbar \hat{X}_{k-1}$

$$\mathbb{D}_{k-2} \quad Y Y Z \cdots Z Z Y$$

$\mathbb{E}_{k-2}$

$$\overset{k-2}{Y} Y Z \cdots Z \overset{2k-4}{X}$$

with $\hat{Y}_{2k-4} \hat{Y}_{2k-3}$

basic relation ⊛ for $\mathbb{D}_{k-2}$ $\Rightarrow$ $-2i\hbar\, q_{\mathbb{C}_{k-2}} - 2i\, q_{\mathbb{E}_{k-2}} = 0$ (2)

basic relations ⊗ for $\mathbb{D}_j$ $(j=1,\cdots,k-2)$

$q_{\mathbb{C}_j} = q$ (1) for all j $\qquad \tilde{q}_j = q_{\mathbb{E}_j}$ (2)

$j=1$ (p13-(1)) $\qquad h q + \tilde{q}_2 = 0$ (3)

even j (p11-(4)) $\qquad h q - \tilde{q}_j - \tilde{q}_{j+1} = 0$ (4)

odd $j \neq 1, k-2$ (p12-(2)) $\qquad h q + \tilde{q}_j + \tilde{q}_{j+1} = 0$ (5)

$j = k-2$ (p13-(1)) $\qquad h q + \tilde{q}_{k-2} = 0$ (6)

summing all these up $\qquad (k-2) h q = 0$ (7)

$k \geq 3,\ h \neq 0 \rightarrow q = 0$ (8) $\qquad \boxed{q_{\mathbb{C}_j} = 0 \text{ for all } j}$ (9)

for standard forms (YY), (XX) with odd k ⇒ **DONE!**

even k $\quad$ similar analysis with $\mathbb{D}_j, \mathbb{E}_j$ $(j=1,\cdots,k-1)$

similar analysis for standard forms (XY), (YX) //

§ notes

- the $S=\frac{1}{2}$ XY (or XX) model $\hat{H}=\sum_{u=1}^{L}(\hat{X}_u\hat{X}_{u+1}+\hat{Y}_u\hat{Y}_{u+1}+h\hat{X}_u)$, $h\neq 0$
possesses no local conserved quantities with length k $(3\leq k\leq \frac{L}{2})$

Yamaguchi, Chiba, Shiraishi: 2024 (based on Shiraishi: 2019)

- all known integrable models have serieses of local conserved quantities

it is strongly suggested that the above model is "non-integrable"

definition ??

extensions of Shiraishi's proof to various quantum spin systems

(most $S=\frac{1}{2}$ models on the d-dim hypercubic lattice with $d\geq 2$ (except for the classical Ising model) do not possess nontrivial local conserved quantities)

$\Downarrow$??

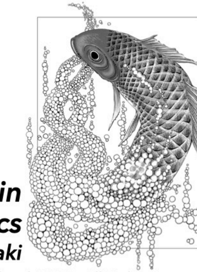
dichotomy: a quantum spin system is either integrable by known methods or does not possess nontrivial local conserved quantities.

we will certainly learn more in the near future!!

relations to integrability, time-evolution, ...

$S=\frac{1}{2}$ XY quantum spin chain

$$\hat{H} = \sum_{u=1}^{L} \left(\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1} + h \hat{X}_u \right) \qquad (1)$$



Advanced Topics in Statistical Physics
by Hal Tasaki

Hal Tasaki (1959–) Elliott Lieb (1932–)

$h=0$ integrable

one can "solve" the model to obtain energy eigenstates and eigenvalues

Lieb, Schultz, Mattis 1961

$h \neq 0$ "non-integrable"

there exist no nontrivial local conserved quantities

it is likely that one can never "solve" the model

Shiraishi 2019

Yamaguchi, Chiba, Shiraishi 2024

Naoto Shiraishi (1989–)