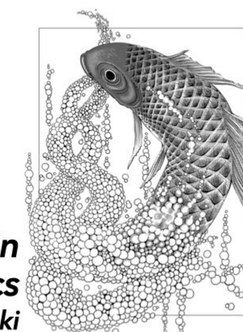
Integrable and non-integrable quantum spin chains

part Ib Jordan-Wigner transformation and the exact solution of the model with $h=0$



Advanced Topics in Statistical Physics
by Hal Tasaki

§ raising and lowering operators

$S=\frac{1}{2}$ quantum spin chain on $\Lambda=\{1,2,\dots,L\}$

$$\hat{H}=\sum_{u\in\Lambda}(\hat{X}_u\hat{X}_{u+1}+\hat{Y}_u\hat{Y}_{u+1})=2\sum_{u\in\Lambda}(\hat{S}_u^+\hat{S}_{u+1}^-+\hat{S}_u^-\hat{S}_{u+1}^+) \quad (1)$$

$$\begin{cases}\hat{S}_u^+=\frac{1}{2}(\hat{X}_u+i\hat{Y}_u) & (2)\\ \hat{S}_u^-=(\hat{S}_u^+)^\dagger=\frac{1}{2}(\hat{X}_u-i\hat{Y}_u) & (3)\end{cases}$$

$$\hat{S}^+=\begin{pmatrix}0&1\\0&0\end{pmatrix} \quad (4) \qquad \hat{S}^-=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad (5)$$

$$(\hat{S}_u^+)^2=(\hat{S}_u^-)^2=0 \quad (6)$$

one can use this

$$\{\hat{S}_u^+,\hat{S}_u^-\}=\hat{S}_u^+\hat{S}_u^-+\hat{S}_u^-\hat{S}_u^+=\frac{1}{4}\{(X+iY)(X-iY)+(X-iY)(X+iY)\}$$
$$=\frac{1}{4}\{X^2+Y^2+iYX-iXY+X^2+Y^2-iYX+iXY\}=1 \quad (7)$$

fermionic ?

$([\hat{X}_u,\hat{X}_v]=[\hat{X}_u,\hat{Y}_v]=\cdots=0)$

for $u\neq v$ $\quad [\hat{S}_u^+,\hat{S}_v^+]=[S_u^+,S_v^-]=0 \quad (8)$

§ Jordan-Wigner transformation (Jordan, Wigner, 1928)

define $\hat{c}_u = \left(\bigotimes_{v=1}^{u-1} \hat{Z}_v\right) \otimes \hat{S}_u^- = \hat{Z}_1 \cdots \hat{Z}_{u-1}\hat{S}_u^-$ (1) $\quad \hat{c}_u^\dagger = \hat{Z}_1\cdots\hat{Z}_{u-1}\hat{S}_u^+$ (2)

$\left(\hat{S}_u^- = \hat{Z}_1\cdots \hat{Z}_{u-1}\hat{c}_u \right.$ (3) $\quad \left. \hat{S}_u^+ = \hat{Z}_1 \cdots \hat{Z}_{u-1} \hat{c}_u^\dagger \right.$ (4))

since $(\hat{Z}_u)^2 = 1$, $\quad \hat{c}_u^{\,2} = \hat{c}_u^{\dagger\,2} = 0$ (5) $\quad \{\hat{c}_u^\dagger, \hat{c}_u\} = \{\hat{S}_u^+, \hat{S}_u^-\} = 1$ (6)

note $\hat{Z}_u \hat{S}_u^- = \hat{Z}_u \frac{1}{2}(\hat{X}_u - i\hat{Y}_u) = \frac{1}{2}(i\hat{Y}_u - i(-i\hat{X}_u)) = -\hat{S}_u^-$ (7)

$\hat{S}_u^- \hat{Z}_u = \hat{S}_u^-$ (8) $\quad \hat{Z}_u \hat{S}_u^+ = \hat{S}_u^+$ (9) $\quad \hat{S}_u^+ \hat{Z}_u = -\hat{S}_u^+$ (10)

for $1\leq u < v \leq L$

$$\hat{c}_u \hat{c}_v = \hat{Z}_1\cdots\hat{Z}_{u-1}\hat{S}_u^- \hat{Z}_1 \cdots \hat{Z}_{u-1}\hat{Z}_u \cdots \hat{Z}_{v-1}\hat{S}_v^- = \hat{S}_u^- \hat{Z}_u \hat{Z}_{u+1}\cdots \hat{Z}_{v-1}\hat{S}_v^- \quad (11)$$

$$\hat{c}_v \hat{c}_u = \hat{Z}_1 \cdots \hat{Z}_{u-1}\hat{Z}_u \hat{Z}_{u+1}\cdots\hat{Z}_{v-1}\hat{S}_v^- \hat{Z}_1\cdots\hat{Z}_{u-1}\hat{S}_u^- = \hat{Z}_u \hat{S}_u^- \hat{Z}_{u+1}\cdots \hat{Z}_{v-1}\hat{S}_v^- \quad (12)$$

$\therefore \{\hat{c}_u, \hat{c}_v\} = 0$ (13) $\quad$ similarly $\{\hat{c}_u, \hat{c}_v^\dagger\} = 0$ (14)

$$\{\hat{c}_u, \hat{c}_v\} = \{\hat{c}_u^\dagger, \hat{c}_v^\dagger\} = 0 \quad (1)$$

$$\{\hat{c}_u, \hat{c}_v^\dagger\} = \delta_{u,v} \quad (2) \qquad \text{for } \forall u, v \in \Lambda$$

canonical anticommutation relations ! part Ia - p4

number operator

$$\hat{n}_u = \hat{c}_u^\dagger \hat{c}_u = \hat{S}_u^+ \hat{S}_u^- = \frac{1}{2}(\hat{X}_u + i\hat{Y}_u)\frac{1}{2}(\hat{X}_u - i\hat{Y}_u) = \frac{1}{2}(1 + \hat{Z}_u) \quad (3)$$

$$n_u = 1 \leftrightarrow Z_u = 1 \qquad n_u = 0 \leftrightarrow Z_u = -1$$

number of fermions = number of up-spins

state $|\Phi_0\rangle$

$$\hat{c}_u|\Phi_0\rangle = \hat{Z}_1 \cdots \hat{Z}_{u-1}\hat{S}_u^-|\Phi_0\rangle = 0 \quad (4) \qquad \text{for } \forall u \in \Lambda$$

$$|\Phi_0\rangle = \bigotimes_{u\in\Lambda} |-\rangle_u = |-, -, \cdots, -\rangle \quad (5)$$

all down state

§ transformation of the Hamiltonian and the exact solution

$$\hat{H} = 2\sum_{u\in\Omega} (\hat{S}^{+}_{u}\hat{S}^{-}_{u+1} + \hat{S}^{-}_{u}\hat{S}^{+}_{u+1}) = \circledast \quad (1)$$

$$\hat{c}^{\dagger}_{u}\hat{c}_{u+1} = \hat{Z}_1\cdots\hat{Z}_{u-1}\hat{S}^{+}_{u}\hat{Z}_1\cdots\hat{Z}_{u-1}\hat{Z}_u\hat{S}^{-}_{u+1} = \underbrace{\hat{S}^{+}_{u}\hat{Z}_u}_{= -\hat{S}^{+}_{u}}\hat{S}^{-}_{u+1} = -\hat{S}^{+}_{u}\hat{S}^{-}_{u+1} \quad (2)$$

$$\hat{c}_{u}\hat{c}^{\dagger}_{u+1} = \hat{Z}_1\cdots\hat{Z}_{u-1}\hat{S}^{-}_{u}\hat{Z}_1\cdots\hat{Z}_u\hat{S}^{+}_{u+1} = \underbrace{\hat{S}^{-}_{u}\hat{Z}_u}_{= \hat{S}^{-}_{u}}\hat{S}^{+}_{u+1} = \hat{S}^{-}_{u}\hat{S}^{+}_{u} \quad (3)$$

$\therefore$

$$\circledast = -\hat{c}^{\dagger}_{u}\hat{c}_{u+1} + \hat{c}_{u}\hat{c}^{\dagger}_{u+1} = -(\hat{c}^{\dagger}_{u}\hat{c}_{u+1} + \hat{c}^{\dagger}_{u+1}\hat{c}_{u}) \quad (4) \leftarrow$$ free fermion Hamiltonian!

BUT $\circledast$ with $L = \hat{S}^{+}_{L}\hat{S}^{-}_{1} + \hat{S}^{-}_{L}\hat{S}^{+}_{1}$ (5)

$$\hat{c}^{\dagger}_{L}\hat{c}_{1} = \hat{Z}_1\cdots\hat{Z}_{L-1}\hat{S}^{+}_{L}\hat{S}^{-}_{1} = \hat{\Pi}\hat{Z}_L\hat{S}^{+}_{L}\hat{S}^{-}_{1} = \hat{\Pi}\hat{S}^{+}_{L}\hat{S}^{-}_{1} \quad (6)$$

$$\hat{c}_{L}\hat{c}^{\dagger}_{1} = -\hat{\Pi}\hat{S}^{-}_{L}\hat{S}^{+}_{1} \quad (7)$$

with parity operator $\hat{\Pi} = \bigotimes_{u\in\Lambda}\hat{Z}_u$ (8)

transformed Hamiltonian depends on $\hat{\Pi}$

decomposition of the Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \quad (1)$$

with $$\mathcal{H}_\pm = \{ |\Phi\rangle \in \mathcal{H} \mid \hat{\Pi} |\Phi\rangle = \pm |\Phi\rangle \} \quad (2)$$

bases of
$$\begin{cases} \mathcal{H}: & |\sigma\rangle = |\sigma_1\rangle_1 \otimes \cdots \otimes |\sigma_L\rangle_L \quad (3) \quad \text{with all } \sigma_1, \cdots, \sigma_L = \pm 1 \\ \mathcal{H}_\pm: & |\sigma\rangle \quad \text{with } \prod_{u=1}^{L} \sigma_u = \pm 1 \quad (4) \end{cases}$$

N: number of fermions = number of + spins

$\mathcal{H}_+$: $L-N$ is even, $\mathcal{H}_-$: $L-N$ is odd energy eigenstates and eigenvalues in $\mathcal{H}_-$ $\boxed{\hat{\Pi} = -1}$

$$\hat{H} = \sum_{u=1}^{L} (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) = -2 \sum_{u=1}^{L} (\hat{c}_u^\dagger \hat{c}_{u+1} + \hat{c}_{u+1}^\dagger \hat{c}_u) \quad (1)$$

$\Downarrow$

the standard free fermion Hamiltonian treated in Ia with $t=2$!

- choose N s.t. $0 \leq N \leq L$ and $L-N$ is odd $\quad$ ($e^{ikL}=1$ for $k \in K$)
- choose $k_1, \dots k_N \in K = \left\{ \frac{2\pi}{L} n \,\middle|\, n = 1, \dots, L \right\}$ (2) s.t. $k_1 < k_2 < \cdots < k_N$

$$\boxed{\begin{aligned} &\text{energy eigenstate } |\Psi_{k_1,\dots,k_N}\rangle = \hat{a}_{k_1}^\dagger \cdots \hat{a}_{k_N}^\dagger |\Phi_0\rangle \quad (3) \\ &\text{energy eigenvalues } E_{k_1,\dots,k_N} = -4 \sum_{j=1}^{N} \cos k_j \quad (4) \end{aligned}}$$

with $|\Phi_0\rangle = \bigotimes_{u \in \Lambda} |-\rangle_u$ (5) $\quad$ all down state

raises the spin at u

$$\hat{a}_k^\dagger = \hat{c}^\dagger(\varphi^{(k)}) = \frac{1}{\sqrt{L}} \sum_{u=1}^{L} e^{iku} \hat{c}_u^\dagger = \frac{1}{\sqrt{L}} \sum_{u=1}^{L} e^{iku} \hat{Z}_1 \cdots \hat{Z}_{u-1} \hat{S}_u^\dagger \quad (6)$$

remark:

Why don't we simply use $\hat{b}_k^\dagger = \frac{1}{\sqrt{L}} \sum_{u=1}^{L} e^{iku} \hat{S}_u^\dagger$ (1) ??

$\hat{b}_k^\dagger |\Phi_0\rangle$ is indeed an energy eigenstate

but $b_{k_1}^\dagger \cdots b_{k_N}^\dagger |\Phi_0\rangle$ is not

$$[\hat{H}, \hat{b}_k^\dagger] = 2 \sum_{u=1}^{L} \left(\hat{Z}_{u+1} e^{ik} + \hat{Z}_{u-1} e^{-ik} \right) \frac{1}{\sqrt{L}} e^{iku} \hat{S}_u^\dagger$$

$$\neq -4 \cos k \, \hat{b}_k^\dagger \quad (2)$$

unwanted phase factors

energy eigenstates and eigenvalues in $\mathcal{H}_+$

$$\hat{H} = \sum_{u=1}^{L} (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) \qquad \boxed{\hat{\Pi} = 1}$$

$$= -2 \sum_{u=1}^{L-1} (\hat{c}_u^\dagger \hat{c}_{u+1} + \hat{c}_{u+1}^\dagger \hat{c}_u) + 2(\hat{c}_L^\dagger \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_L) = \hat{B}(\tilde{T}) \quad (1)$$

single-particle Schrödinger equation

$$-t(\varphi_{u-1} + \varphi_{u+1}) = \epsilon \varphi_u \quad (u = 2, \cdots, L-1), \quad -t(-\varphi_L + \varphi_2) = \epsilon \varphi_1, \quad -t(\varphi_{L-1} - \varphi_1) = \epsilon \varphi_L \quad (2)$$

$$\psi_u^{(k)} = \frac{1}{\sqrt{L}} e^{iku} \quad (3) \quad \text{with } k \in \tilde{K} = \left\{ \frac{2\pi}{L}\left(n + \frac{1}{2}\right) \,\middle|\, n = 1, \cdots, L \right\} \quad (4)$$

$$\left(e^{ikL} = -1 \text{ for } k \in \tilde{K} \right)$$

- choose N s.t. $0 \leq N \leq L$ and $L - N$ is even
- choose $k_1, \cdots k_N \in \tilde{K}$ s.t. $k_1 < k_2 < \cdots < k_N$

energy eigenstate $|\Psi_{k_1, \cdots, k_N}\rangle = \hat{a}_{k_1}^\dagger \cdots \hat{a}_{k_N}^\dagger |\Phi_0\rangle$ (5)

energy eigenvalues $E_{k_1, \cdots, k_N} = -4 \sum_{j=1}^{N} \cos k_j$ (6)

§ conserved quantities

$$[\hat{H}, \hat{Q}] = 0 \quad (1) \rightarrow \hat{Q} \text{ is a conserved quantity}$$

$$\hat{Q}_1 = \sum_{u=1}^{L} \hat{Z}_u \quad (2) \qquad \hat{Q}_2 = \hat{H} = \sum_{u=1}^{L} (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) \quad (3)$$

systematic construction

S: $L \times L$ matrix s.t. $[T, S] = 0$ (4) $\qquad \hat{Q} = \hat{B}(S)$ (5)

$$[\hat{H}, \hat{Q}] = [\hat{B}(T), \hat{B}(S)] = \hat{B}([T, S]) = 0 \quad (6)$$

(part Ia p.7)

examples

$$[T, T^n] = 0 \quad (7)$$

$$\hat{B}(T^2 - 2t^2 I) = t^2 \sum_{u=1}^{L} (\hat{C}_{u+2}^{\dagger} \hat{C}_u + \hat{C}_u^{\dagger} C_{u+2})$$

$$= -\frac{t^2}{2} \sum_{u=1}^{L} (\hat{X}_u \hat{Z}_{u+1} \hat{X}_{u+2} + \hat{Y}_u \hat{Z}_{u+1} \hat{Y}_{u+2}) \quad (8)$$

$$\hat{Q}_3 = \sum_{u=1}^{L} (\hat{X}_u \hat{Z}_{u+1} \hat{X}_{u+2} + \hat{Y}_u \hat{Z}_{u+1} \hat{Y}_{u+2}) \quad (9)$$

10

$$(R)_{uv} = \begin{cases} 1 & u = v+1 \\ -1 & u = v-1 \\ 0 & \text{otherwise} \end{cases} \leftarrow \text{p.b.c.} \quad (1)$$

$$[T, R] = 0 \quad (2)$$

$$\hat{B}(R) = \sum_{u=1}^{L} \left(\hat{C}^{\dagger}_{u+1} \hat{C}_u - \hat{C}^{\dagger}_u \hat{C}_{u+1} \right) = -\frac{i}{2} \sum_{u=1}^{L} \left(\hat{X}_u \hat{Y}_{u+1} - \hat{Y}_u \hat{X}_{u+1} \right) \quad (3)$$

$$\hat{Q}'_2 = \sum_{u=1}^{L} \left(\hat{X}_u \hat{Y}_{u+1} - \hat{Y}_u \hat{X}_{u+1} \right) \quad (4)$$

more conserved quantities from $\hat{B}(T^n)$ and $\hat{B}(T^n R)$

decomposition of the Hilbert space? $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

the construction was for $\mathcal{H}_-$, but $\hat{Q}_2, \hat{Q}_3, \cdots, \hat{Q}'_2, \cdots$ are consered quantities on the whole Hilbert space $\mathcal{H}$!

why?

- one can explicitly check $[\hat{H}, \hat{Q}] = 0$
- $\hat{B}(\tilde{T}^n)$, $\hat{B}(\tilde{T}^n \tilde{R})$ produce the same quantities!

§ notes

the model was solved by mapping it to a free fermion model by the Jordan-Wigner transformation

→ engery eigenstates, energy eigenvalues, free energy, conserved quantities, …

with the same technique one can also solve the models

$$\hat{H} = \sum_{u=1}^{L} \{ J \hat{X}_u \hat{X}_{u+1} + J' \hat{Y}_u \hat{Y}_{u+1} + h \hat{Z}_u \}$$ XY model

$$\hat{H} = \sum_{u=1}^{L} \{ \hat{Z}_u \hat{Z}_{u+1} + h \hat{X}_u \}$$ Ising model under transverse magnetic field

with a more sophisticated Bethe ansatz technique, one can solve

$$\hat{H} = \sum_{u=1}^{L} \{ J (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) + J' \hat{Z}_u \hat{Z}_{u+1} + h \hat{Z}_u \}$$ XXZ model

$$\hat{H} = \sum_{u=1}^{L} \{ J \hat{X}_u \hat{X}_{u+1} + J' \hat{Y}_u \hat{Y}_{u+1} + J'' \hat{Z}_u \hat{Z}_{u+1} \}$$ XYZ model

see, e.g., the review:
F. Franchini, arXiv:1609.02100

decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (1) 5

$$\mathcal{H}_\pm = \{ |\Phi\rangle \in \mathcal{H} \mid \hat{\Pi} |\Phi\rangle = \pm |\Phi\rangle \} \quad (2)$$

bases of $\mathcal{H}_\pm$ $\quad |\sigma\rangle = |\sigma_1\rangle_1 \otimes \cdots \otimes |\sigma_L\rangle_L$ (3) with $\prod_{u=1}^{L} \sigma_u = \pm 1$ (4)

on $\mathcal{H}_-$ $\quad \hat{\Pi} = -1 \rightarrow$ number of $\sigma_u = -1$ is odd $\quad L - N$ is odd

$$\hat{H} = \sum_{u=1}^{L} (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) = -2 \sum_{u=1}^{L} (\hat{c}_u^\dagger \hat{c}_{u+1} + \hat{c}_{u+1}^\dagger \hat{c}_u) \quad (5)$$

N s.t. $L-N$ is odd $(0 \leq N \leq L)$ $\qquad$ standard free fermion Hamiltonian!

$k_1, \ldots, k_N \in \mathcal{K}$ with $0 < k_1 < \cdots < k_N \leq 2\pi$

energy e.s. $\hat{a}_{k_1}^\dagger \cdots \hat{a}_{k_N}^\dagger |\Phi_0\rangle$ (6) $\quad$ energy $E_{k_1, \ldots, k_N} = -4 \sum_{j=1}^{N} \cos k_j$ (7)

on $\mathcal{H}_+$ $\quad \hat{\Pi} = 1$ $\qquad$ one can also solve this

$$\hat{H} = -2 \sum_{u=1}^{L-1} (\hat{c}_u^\dagger \hat{c}_{u+1} + \hat{c}_{u+1}^\dagger \hat{c}_u) + 2 (\hat{c}_L^\dagger \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_L) \quad (8)$$

§ conserved quantities

$[\hat{H}, \hat{Q}] = 0$ (1) $\hat{Q}$ is a conserved quantity

S: $L \times L$ matrix s.t. $[T, S] = 0$ (2) part Ia p.7

$$[\hat{H}, \hat{B}(S)] = [\hat{B}(T), \hat{B}(S)] = \hat{B}([T, S]) = 0 \quad (3)$$

$\hat{Q} = \hat{B}(S)$ is a conserved quantity

example
$$\hat{B}(T^2) = t^2 \sum_{u=1}^{L} (2\hat{C}_u^\dagger \hat{C}_u + \hat{C}_{u+2}^\dagger \hat{C}_u + \hat{C}_u^\dagger \hat{C}_{u+2})$$
$$= -\frac{t^2}{2} \sum_{u=1}^{L} (\hat{X}_u \hat{Z}_{u+1} \hat{X}_{u+2} + \hat{Y}_u \hat{Z}_{u+1} \hat{Y}_{u+2} - 2\hat{Z}_u - 2) \quad (4)$$
$$\hat{Q} = \sum_{u=1}^{L} (\hat{X}_u \hat{Z}_{u+1} \hat{X}_{u+2} + \hat{Y}_u \hat{Z}_{u+1} \hat{Y}_{u+2} - 2\hat{Z}_u) \quad (5)$$

p.b.c.

$$(\tilde{T})_{uv} = \begin{cases} 1 & u = v+1 \\ -1 & u = v-1 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

- other examples $\hat{B}(T^n)$, $\hat{B}(T^n \tilde{T})$
- what happenes for the subspace $\mathcal{H}_+$

§ remarks

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→ exact engery eigenstate, energy eigenvalues, free energy, ...

with the same technique one can also solve the models

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with a more sophisticated Bethe ansatz technique, one can solve

$$\hat{H} = \sum_{u=1}^{L} \{ J(\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1}) + J' \hat{Z}_u \hat{Z}_{u+1} + h \hat{Z}_u \}$$ XXZ model

$$\hat{H} = \sum_{u=1}^{L} \{ J \hat{X}_u \hat{X}_{u+1} + J' \hat{Y}_u \hat{Y}_{u+1} + J'' \hat{Z}_u \hat{Z}_{u+1} \}$$ XYZ model