Integrable and non-integrable quantum spin chains

Advanced Topics in Statistical Physics
by Hal Tasaki

$S=\frac{1}{2}$ XY quantum spin chain

$$\hat{H} = \sum_{u=1}^{L} \left(\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1} + h \hat{X}_u \right) \quad (1)$$

$h=0$ integrable

one can "solve" the model to obtain
energy eigenstates and eigenvalues

Lieb, Schultz, Mattis 1961

$h \neq 0$ "non-integrable"

there exist no nontrivial local conserved quantities
it is likely that one can never "solve" the model

Shiraishi 2019

Yamaguchi, Chiba, Shiraishi 2024

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Hal Tasaki (1959–) Elliott Lieb (1932–)

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Shiraishi 2019

Yamaguchi, Chiba, Shiraishi 2024

Naoto Shiraishi (1989–)

§ definition of the model

single spin with $S=\frac{1}{2}$

basis states $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Pauli operators $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

any operator can be expanded as $a_0 \hat{I} + a_1 \hat{X} + a_2 \hat{Y} + a_3 \hat{Z}$

properties of the Pauli matrices

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$

$$\hat{X}\hat{Y} = -\hat{Y}\hat{X} = i\hat{Z}, \quad \hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = i\hat{X}, \quad \hat{Z}\hat{X} = -\hat{X}\hat{Z} = i\hat{Y}$$

$$[\hat{X}, \hat{Y}] = 2i\hat{Z}, \quad [\hat{Y}, \hat{Z}] = 2i\hat{X}, \quad [\hat{Z}, \hat{X}] = 2i\hat{Y}$$

$$\{\hat{X}, \hat{Y}\} = \{\hat{Y}, \hat{Z}\} = \{\hat{Z}, \hat{X}\} = 0$$

$$\left(\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \right)$$

quantum spin chain

spin on each site u of the chain $\Lambda = \{1, 2, \cdots, L\}$

basis states at $u \in \Lambda$ $\quad |+\rangle_u, \ |-\rangle_u$ $\quad$ 2^L states

basis states of the whole system $\quad |\boldsymbol{\sigma}\rangle = |\sigma_1\rangle_1 \otimes |\sigma_2\rangle_2 \otimes \cdots \otimes |\sigma_L\rangle_L$

spin configuration $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_L)$, $\quad \sigma_u = \pm 1$

$\hat{X}_u, \hat{Y}_u, \hat{Z}_u$ copies of $\hat{X}, \hat{Y}, \hat{Z}$ at site u

$$\hat{X}_u |\boldsymbol{\sigma}\rangle = |\sigma_1\rangle_1 \otimes \cdots \otimes |\sigma_{u-1}\rangle_{u-1} \otimes \hat{X}_u |\sigma_u\rangle_u \otimes |\sigma_{u+1}\rangle_{u+1} \otimes \cdots \otimes |\sigma_L\rangle_L$$

acts only here

Hamiltonian of the XY model (or the XX model)

$$\hat{H} = \sum_{u=1}^{L} (\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1} + h \hat{X}_u)$$

periodic boundary conditions $\quad \hat{X}_{L+1} = \hat{X}_1, \ \hat{Y}_{L+1} = \hat{Y}_1$

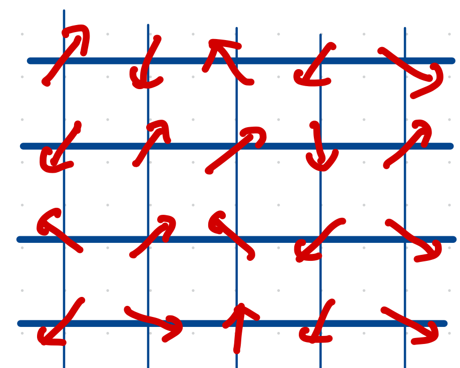
§ about the model

more realistic model (XXZ model)

$$\hat{H} = \sum_{u} \left\{ J\left(\hat{X}_u \hat{X}_{u+1} + \hat{Y}_u \hat{Y}_{u+1} + \lambda \hat{Z}_u \hat{Z}_{u+1}\right) - \frac{\mu_0}{2}\left(h_x X_u + h_y Y_u + h_z Z_u\right) \right\}$$

small λ → XX model

much more realistic model (quasi one dimensional model)



$$\hat{H} = \sum_{u} J\left(\hat{X}_u \hat{X}_{u+e_1} + \hat{Y}_u \hat{Y}_{u+e_1} + \lambda \hat{Z}_u \hat{Z}_{u+e_1}\right) + \sum_{\substack{u \\ \alpha=2,3}} J'\left(\hat{X}_u \hat{X}_{u+e_\alpha} + \hat{Y}_u \hat{Y}_{u+e_\alpha} + \lambda' \hat{Z}_u \hat{Z}_{u+e_\alpha}\right) - \frac{\mu_0}{2} \sum_{u}\left(h_x \hat{X}_u + h_y \hat{Y}_u + h_z \hat{Z}_u\right)$$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

small J' → collection of independent quantum spin chains

§plan

part Ia free fermions on the chain

part Ib Jordan-Wigner transformation and the exact solution of the model with $h=0$

part II proof of the absence of nontrivial local conserved quantities in the model with $h\neq 0$