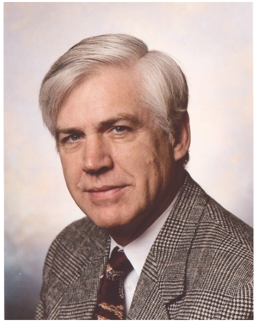

Thomas Spencer (1946-)

The absence of ferromagnetic order in the two-dimensional XY model

part 4 Mcbryan-Spencer's proof of the absence of order

Advanced Topics in Statistical Physics
by Hal Tasaki

The picture of Spencer from the web page of IAS.

§ bound on the correlation function for the XY model in $d=2$ with $h=0$

$$\Lambda_L = \{1,\dots,L\}^2, \quad \mathcal{B}_L = \{\{u,v\} \mid u \text{ and } v \text{ are nearest neighbors (periodic b.c.)}\} \tag{1}$$

$$\Theta = (\theta_u)_{u\in\Lambda_L}, \quad \theta_u \in [0,2\pi), \quad \int d\Theta = \prod_{u\in\Lambda_L} \int_0^{2\pi} d\theta_u \tag{2}$$

$$H_L(\Theta) = -\sum_{\{u,v\}\in\mathcal{B}_L} \cos(\theta_u - \theta_v) \tag{3}$$

$$Z_L(\beta) = \int d\Theta \, e^{-\beta H_L(\Theta)} \tag{4}$$

$$\langle \cdots \rangle_{L,\beta} = \frac{1}{Z_L(\beta)} \int d\Theta \, (\cdots)\, e^{-\beta H_L(\Theta)} \tag{5}$$

proved in part 5

theorem we have for any $0<\beta<\infty$ that

$$0 \le \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} = \langle e^{i(\theta_u-\theta_v)} \rangle_{L,\beta} \le |u-v|^{-\eta(\beta)} \tag{6}$$

for any $u,v\in\Lambda_L$ s.t. $|u-v| \le \frac{L}{2}$ with $\eta(\beta)>0$

we also have $\eta(\beta) \simeq (2\beta C)^{-1}$ if $\beta \gg 1$ (C is a constant)

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Communications in Mathematical Physics

On the Decay of Correlations in $SO(n)$-symmetric Ferromagnets*

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Abstract. We prove that for low temperatures T the spin-spin correlation function of the two-dimensional classical $SO(n)$-symmetric Ising ferromagnet decays faster than $|x|^{-\text{const}\, T}$ provided $n \geqq 2$. We also discuss a nearest neighbor continuous spin model, with spins restricted to a finite interval, where we show that the spin-spin correlation function decays exponentially in any number of dimensions.

I. Introduction and Results

The Mermin-Wagner theorem [1] states that at non-zero temperatures the two dimensional Heisenberg model has no spontaneous magnetization. Consequently the spin-spin correlation function decays to zero at large distances, although the Mermin-Wagner theorem gives no indication of the rate of decay. Similar results apply for the classical $SO(n)$-symmetric ($n \geqq 2$) nearest neighbor Ising ferromagnets which we study here, see for example the paper of Mermin [2]. We establish a polynomial upper bound for the decay rate of the spin-spin correlation function for these models at very low temperatures. Fisher and Jasnow [3] have previously obtained a $\log^{-1}|x|$ decay.

To describe the $SO(n)$-symmetric ferromagnet, we consider the infinite lattice of unit spacing with sites labelled by indices $i \in \mathbb{Z}^2$. To each site i we associate an n-component classical spin s_i of unit length, $\|s_i\| = 1$. The spin-spin correlation function at inverse temperature $\beta = T^{-1}$ is

$$\langle s_0 \cdot s_x \rangle(\beta) = Z^{-1} \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j} s_0 \cdot s_x \,, \qquad Z = \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j} \,, \tag{1}$$

where $\sum_{\langle i,j \rangle}$ denotes a sum over nearest neighbor pairs, $\Omega_i^{(n)}$ is the invariant measure

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ons and (1) is to be interpreted as the thermodynamic
olume quantities $\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (1) but
$+1) \times (2N+1)$ periodic lattice. Let $C(x)$ denote the
ice's equation on the lattice: $-\Delta C(x) = \delta_{0,x}$, $C(0) = 0$.

$\beta \geqq \beta_0(\varepsilon)$ *sufficiently large*

(2)

-spin correlation functions are known to decay

le case $n=2$ in Section II; other values of n are
neralization. For $n=2$ we use the representation s
the form

$\sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)$ $\cos(\phi_0 - \phi_x)$. (3)

vated by the approximation [4]

$(\phi_i - \phi_j)^2 = \frac{1}{2}(\phi \Delta \phi)$. (4)

ving the limit of integration in (3) to extend to
correlation

$d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \cos(\phi_0 - \phi_x) \Big/ \left(\prod_i \int_{-\infty}^{\infty} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \right)$, (5)

Theorem 1. It is difficult to justify the two
use (3) is the integral of a periodic function of the
be extended to infinity without changing (3),
on (4) is then unreasonable, since it makes sense
below) we show that there is a marked difference
) depending on whether the integration range is
orrelations on a ν-dimensional lattice by

$\phi_0 \phi_x \Big/ \left(\prod_i \int_{-\mu}^{\mu} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \right)$ (6)

te μ, β, ν *there is an* $\tilde{m} > 0$ *such that*
$t\, e^{-\tilde{m}|x|}$, $|x| \to \infty$. (7)

y be chosen at least as large as $\cosh^{-1}(1 + m^2/4)$, m
contrast to the exponential decay for finite μ, (5) always
for $\mu = \infty$ in two dimensions.

nd for the Plane Rotator

se the representation (3) for $n=2$, replacing $\cos(\phi_0 - \phi_x)$
$-\phi_x)\rangle(\beta) = 0$. Using the periodicity of the integrand, we

$^{1}(C(j) - C(j-x))$,

s means that we deform the path of integration and use
to cancel the lateral contours. The above translation
$e^{iz}| = 1$, z real, yields

$\prod_i \int d\phi_i e^{\beta \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)\cosh(a_i - a_j)}$

$\sum_{\langle i,j \rangle} (\cosh(a_i - a_j) - 1)$

amental solution $C(x)$ we prove below that
$= 1$, uniformly in x, i, j. (8)
d $\beta_0(\varepsilon)$ such that for $\beta \geqq \beta_0$

$+\varepsilon) \sum_{\langle i,j \rangle} (a_i - a_j)^2 = \frac{1}{2}(1+\varepsilon)(a, -\Delta a)$
$= (1+\varepsilon)(2\beta)^{-1}(a_0 - a_x)$. (9)

$C(x)$ we obtain the bound of Theorem 1 from (7) and

s been formal in that (3) should be interpreted as a
$\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (3) but with sites in a $(2N$
. All of the steps above are valid for $\langle s_0 \cdot s_x \rangle(\beta, N)$
where by the corresponding fundamental solution

$x - 1)/(4 - 2\cos k_1 - 2\cos k_2)$, (10)

$+1)^{-1} r_i$, r_i integers, $|r_i| \leqq N\}$. To prove (8) we use
$^{-2}\theta^2$, $|\theta| \leqq \pi$, to obtain for nearest neighbors i, j:

$\sum_{L^*, \neq 0} |k_1|/(k_1^2 + k_2^2) < 2$.

y, uniformly in x, i, j, N.

r $n > 2$, we parameterize the n-sphere by angles
$\leqq \pi$ in such a way that only the components $s^{(1)}$, $s^{(2)}$
ϕ. We then treat $\langle s_0^{(1)} s_x^{(1)} + s_0^{(2)} s_x^{(2)} \rangle = (2/n)\langle s_0 \cdot s_x \rangle$ as
only the variables ϕ_i. Alternatively one may apply
compare $N=2$ with $N \geqq 3$. See [5, 6].

or the Square Well Model

ge of variable $\phi \to \phi/\mu$ reduces the problem to the
$\rangle_{-\Delta,1}(\beta)$ by $\langle \phi_0 \phi_x \rangle$ and note that it is the limit as
xpectations $\langle \phi_0 \phi_x \rangle_p$ defined by replacing $\int_{-1}^{1} d\phi_i$ at

Jsing integration by parts we have for any $m > 0$

$p-1)\langle \phi_0 \phi_x^{p-1} \rangle_p - \beta m \langle \phi_0 \phi_x \rangle_p\}$.

$^{\tilde{m}|x-y|}$, $\tilde{m} \equiv \cosh^{-1}(1 + m^2/4)$,
ind $m > 0$ such that
$, \geqq 0$, all p.
t side of (11) is positive if (11)

o eliminate the ferromagnetic couplings:
$^{\phi^2} e^{-\phi p} \phi^{p-2} / \int d\phi e^{-\nu\beta\phi^2} e^{-\phi p}$ (12)
$d\phi \phi^{p-2} \Big/ \int_{-\infty}^{\infty} d\phi e^{-\nu\beta\phi^2}$
β.

tisfies (12) for all p.
shown using inequalities for log concave
g range order, see [8].

s 17, 1133 (1960)
70)
1971)
67)
5)
(1976)

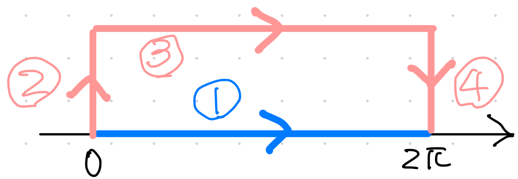
lemma let $f(z)$ be analytic in $z\in\mathbb{C}$ and satisfies $f(z+2\pi)=f(z)$

then $\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} f(\theta+i\varphi)\,d\theta$ (1) for any $\varphi\in\mathbb{R}$

proof

$$\int_0^{2\pi} f(\theta)\,d\theta = \overset{\textcircled{1}}{}\, i\int_0^{\varphi} f(iy)\,dy\ \overset{\textcircled{2}}{} + \int_0^{2\pi} f(\theta+i\varphi)\,d\theta\ \overset{\textcircled{3}}{} - \boxed{i\int_0^{\varphi} f(iy+2\pi)\,dy}\ \overset{\textcircled{4}}{} \qquad (2)$$

$$= i\int_0^{\varphi} f(iy)\,dy$$



thus

$$Z_L(\beta)\langle \vec{S}_u\cdot\vec{S}_v\rangle_{L,\beta} = Z_L(\beta)\langle e^{i(\theta_u-\theta_v)}\rangle_{L,\beta}$$

$$= \int d\Theta\, e^{i(\theta_u-\theta_v)+\beta\sum_{\{w,w'\}\in B_L}\cos(\theta_w-\theta_{w'})}$$

$$\boxed{\theta_w \to \theta_w + i\varphi_w}$$

$$= \int d\Theta\, e^{i(\theta_u+i\varphi_u-\theta_v-i\varphi_v)+\beta\sum_{\{w,w'\}\in B_L}\cos(\theta_w+i\varphi_w-\theta_{w'}-i\varphi_{w'})} \qquad (3)$$

for any $\varphi_w\in\mathbb{R}$ $(w\in\Lambda_L)$

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since $\cos(\theta + i\varphi) = \cos\theta \cosh\varphi - i \sin\theta \sinh\varphi$

$$Z_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta}$$

$$= \int d\Theta \, e^{i(\theta_u + i\varphi_u - \theta_v - i\varphi_v) + \beta \sum_{\{w,w'\} \in B_L} \cos(\theta_w + i\varphi_w - \theta_{w'} - i\varphi_{w'})}$$

$$= \underbrace{e^{-\varphi_u + \varphi_v}}_{\text{nonnegative}} \int d\Theta \left[e^{i\overbrace{\{\theta_u - \theta_v - \beta \sum_{\{w,w'\}} \sin(\theta_w - \theta_{w'}) \sinh(\varphi_w - \varphi_{w'})\}}^{\text{real}}} \times \underbrace{e^{\beta \sum_{\{w,w'\}} \cos(\theta_w - \theta_{w'}) \cosh(\varphi_w - \varphi_{w'})}}_{\text{nonnegative}} \right] \quad (2)$$

$$Z_L(\beta) \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \le e^{-\varphi_u + \varphi_v} \int d\Theta \, e^{\beta \sum_{\{w,w'\}} \cos(\theta_w - \theta_{w'}) \cosh(\varphi_w - \varphi_{w'})}$$

$$\boxed{\cos\theta \cosh\varphi = \cos\theta(\cosh\varphi - 1) + \cos\theta \le (\cosh\varphi - 1) + \cos\theta}$$

$$\le e^{-\varphi_u + \varphi_v + \beta \sum_{\{w,w'\} \in B_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}} \underbrace{\int d\Theta \, e^{\beta \sum_{\{w,w'\} \in B_L} \cos(\theta_w - \theta_{w'})}}_{= Z_L(\beta)} \quad (3)$$

$$\boxed{\text{lemma (McByran-Spencer) for any } \varphi_w \in \mathbb{R}\ (w \in \Lambda_L), \text{ we have} \quad \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \le e^{-\varphi_u + \varphi_v + \beta \sum_{\{w,w'\} \in \mathcal{B}_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}} \quad (1)}$$

proved only by the change of variables!!

we shall chose proper φ_w

$$(\text{RHS of } (1)) \simeq e^{-A(\boldsymbol{\varphi})} \qquad \boldsymbol{\varphi} = (\varphi_w)_{w \in \Lambda_L}$$

$$\text{with } A(\boldsymbol{\varphi}) = \varphi_u - \varphi_v - \frac{\beta}{2} \sum_{\{w,w'\} \in \mathcal{B}_L} (\varphi_w - \varphi_{w'})^2 \qquad (2)$$

we can try maximizing $A(\boldsymbol{\varphi})$

$$\frac{\partial}{\partial \varphi_w} A(\boldsymbol{\varphi}) = \delta_{uw} - \delta_{vw} - \beta \boxed{\sum_{w' \in \mathcal{N}(w)} (\varphi_w - \varphi_{w'})}$$

part3-p4-(5)

$$= \delta_{uw} - \delta_{vw} + \beta \boxed{\sum_{w' \in \Lambda_L} \Delta_{ww'} \varphi_{w'}} \qquad (3)$$

$$\frac{\partial}{\partial \varphi_w} A(\varphi) = 0 \quad (1) \Rightarrow -\sum_{w' \in \Lambda_L} \Delta_{ww'} \varphi_{w'} = \frac{1}{\beta}(\delta_{uw} - \delta_{vw}) \quad (2)$$

Poisson equation! part3-p11-(1)

if we choose φ_w as the solution of (2)

$$A(\varphi) = \varphi_u - \varphi_v - \frac{\beta}{2} \sum_{\{w,w'\} \in \mathcal{B}_L} (\varphi_w - \varphi_{w'})^2 = \varphi_u - \varphi_v + \frac{\beta}{2} \sum_{\{w,w'\} \in \mathcal{B}_L} \varphi_w \Delta_{ww'} \varphi_{w'}$$

$$= \frac{1}{2}\{\varphi_u - \varphi_v\} \quad (3)$$

P5-(1) then implies

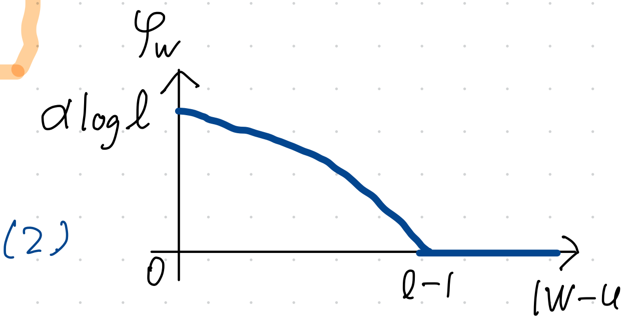
$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{\beta,L} \lesssim \boxed{e^{-\frac{1}{2}\{\varphi_u - \varphi_v\}}} \quad (4)$$

recovers part3-p10-(5) from the harmonic approximation!

☞ we use simpler and explicit choice of φ_w Picco (1984) for $\alpha > 0$, we set

$$\varphi_w = \begin{cases} \alpha \log \dfrac{\ell}{|w-u|+1}, & |w-u| \leq \ell - 1 \\ 0, & |w-u| \geq \ell - 1 \end{cases} \qquad (1)$$

with $\ell = |u-v|$



$$\varphi_u = \alpha \log \ell, \quad \varphi_v = 0 \qquad (2)$$

we can show

$$\sum_{\{w,w'\}\in\mathcal{B}_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} \leq C(\cosh\alpha - 1)\log \ell \qquad (3)$$

with constant $C > 0$

we then get

P5-(1)

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \leq e^{-\varphi_u + \varphi_v + \beta \sum_{\{w,w'\}\in\mathcal{B}_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\}}$$

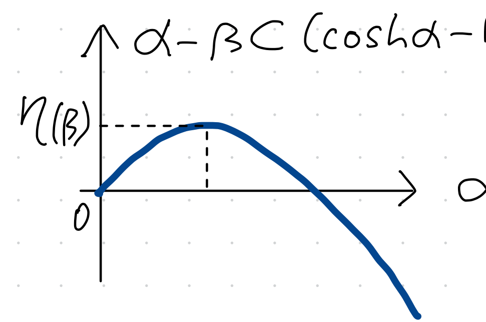
$$\leq e^{-\alpha \log \ell + \beta C(\cosh\alpha - 1)\log \ell}$$

$$= \ell^{-\{\alpha - \beta C(\cosh\alpha - 1)\}} \qquad (4)$$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \le e^{-\{\alpha - \beta C(\cosh\alpha - 1)\}} \quad (1)$$

let

$$\eta(\beta) = \max_{\alpha} \{\alpha - \beta C(\cosh\alpha - 1)\} > 0 \quad (2)$$



$$\boxed{\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta} \le |u-v|^{-\eta(\beta)}} \quad (3)$$

← the main result!

max is attained when $\sinh\alpha = \frac{1}{\beta C}$

$$\eta(\beta) = \sinh^{-1}\left(\frac{1}{\beta C}\right) - \beta C\left\{\sqrt{1 + \frac{1}{(\beta C)^2}} - 1\right\} \simeq \frac{1}{2\beta C} \quad (5)$$

for $\beta C \gg 1$

ⅲ) derivation of p7-(3)

$$\sum_{\{w,w'\}\in \mathcal{B}_L} \{\cosh(\varphi_w-\varphi_{w'})-1\} \leq C(\cosh\alpha -1)\log l \quad (1)$$

for $\{w,w'\}\in\mathcal{B}_L$ s.t. $|w-u|\leq|w'-u|\leq l$ (2)

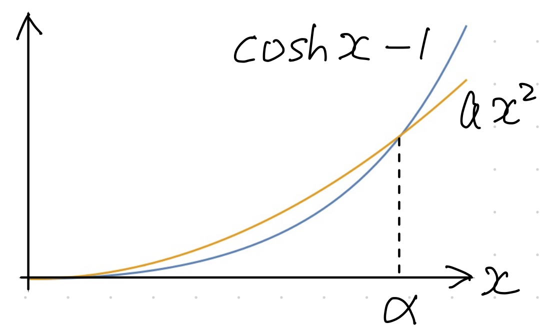
$$|\varphi_w-\varphi_{w'}| = \alpha\log\frac{|w'-u|+1}{|w-u|+1} \leq \alpha\log\frac{|w-u|+1+1}{|w-u|+1}$$

p7-(1)

$$=\alpha\log\left(1+\frac{1}{|w-u|+1}\right)\leq\frac{\alpha}{|w-u|+1} \quad (3)$$

note $\cosh x -1 \leq \dfrac{\cosh\alpha-1}{\alpha^2}x^2$ (4)

for any $x\in[0,\alpha]$



since $|\varphi_w-\varphi_{w'}|\leq\alpha$ (5)

$$\cosh(\varphi_w-\varphi_{w'})-1 \leq \frac{\cosh\alpha-1}{\alpha^2}(\varphi_w-\varphi_{w'})^2 \leq \frac{\cosh\alpha-1}{(|w-u|+1)^2} \quad (6)$$

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$$\sum_{\{w,w'\}\in B_L}' \{\cosh(\varphi_w-\varphi_{w'})-1\} \le 4\,(\cosh\alpha-1)\sum_{\substack{w\in\Lambda_L\\(|w-u|\le\ell)}}' \frac{1}{(|w-u|+1)^2} \qquad (1)$$

choice of w'

$$\mathbf{r}=(r_1,r_2)$$

$$\sum_{\substack{w\in\Lambda_L\\(|w-u|\le\ell)}}' \frac{1}{(|w-u|+1)^2} \le C'\int_{|\mathbf{r}|\le\ell} \frac{1}{(|\mathbf{r}|+C'')^2}\,d\mathbf{r}$$

$$=C'\int_0^{\ell}\frac{2\pi r}{(r+C'')^2}\,dr = \frac{C}{4}\log\ell \qquad (2)$$

$$\sum_{\{w,w'\}\in B_L}' \{\cosh(\varphi_w-\varphi_{w'})-1\} \le C(\cosh\alpha-1)\log\ell \qquad (3)$$

§ proof of the Hohenberg-Mermin-Wagner theorem based on the McBryan-Spencer argument

Tasaki (2020)

the two-dimensional XY model under magnetic field h

$$H_{L,h}(\Theta) = -\sum_{\{u,v\}\in B_L} \cos(\theta_u - \theta_v) - h \sum_{u\in\Lambda_L} \cos\theta_u \qquad (1)$$

$$Z_L(\beta,h) = \int d\Theta \, e^{-\beta H_{L,h}(\Theta)} \qquad (2)$$

$$\langle \cdots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta,h)} \int d\Theta \, (\cdots)\, e^{-\beta H_{L,h}(\Theta)} \qquad (3)$$

theorem for any $0<\beta<\infty$

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty} \Big\langle \frac{1}{L^d}\sum_{u\in\Lambda_L} S_u^{(x)} \Big\rangle_{L,\beta,h} = \lim_{h\downarrow 0}\lim_{L\uparrow\infty} \langle S_u^{(x)} \rangle_{L,\beta,h}$$

$$= \lim_{h\downarrow 0}\lim_{L\uparrow\infty} \langle e^{i\theta_u} \rangle_{L,\beta,h} = 0 \qquad (4)$$

repeating the argument in p3, p4

$$\boxed{|\langle e^{i\theta_u}\rangle_{L,\beta,h}| \leq e^{-\varphi_u + \beta \sum'_{\{w,w'\}\in B_L} \{\cosh(\varphi_w - \varphi_{w'}) - 1\} + \beta h \sum'_{w\in\Lambda_L} (\cosh \varphi_w - 1)}} \quad (1)$$

corresponds to P5-(1)

as in P7-(1)
$$\varphi_w = \begin{cases} \alpha \log \dfrac{\ell}{|w-u|+1}, & |w-u| \leq \ell - 1 \\ 0, & |w-u| \geq \ell - 1 \end{cases} \quad (2)$$

with $0 < \ell \leq \frac{L}{2}$

$$|\langle S_u^{(x)}\rangle_{L,\beta,h}| \leq e^{-\eta(\beta)\log \ell + h\, G(\beta,\ell)} \quad (3)$$

with
$$G(\beta,\ell) = \beta \sum_{w\in\Lambda_L}{}' (\cosh \varphi_w - 1)$$
$$= \beta \sum_{\substack{w\in\Lambda_L \\ (|w-u|\leq \ell)}}{}' \left\{\cosh\left(\alpha(\beta) \log \frac{\ell}{|w-u|+1}\right) - 1\right\} \quad (4)$$

$$\left|\langle S_u^{(x)}\rangle_{L,\beta,h}\right| \leq e^{-\eta(\beta)\log \ell + h\, G(\beta,\ell)} \quad (1)$$

take any $\ell > 0$

for $\begin{cases} L \geq 2\ell \\ h > 0 \text{ s.t. } h\, G(\beta,\ell) \leq \frac{1}{2}\eta(\beta)\log \ell \quad (2) \end{cases}$

we have $$\left|\langle S_u^{(x)}\rangle_{L,\beta,h}\right| \leq e^{-\frac{1}{2}\eta(\beta)\log \ell} \quad (3)$$

thus
$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\left|\langle S_u^{(x)}\rangle_{L,\beta,h}\right| \leq e^{-\frac{1}{2}\eta(\beta)\log \ell} \quad (4)$$

since ℓ is arbitrary $$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\left|\langle S_u^{(x)}\rangle_{L,\beta,h}\right| = 0 \quad (5)$$