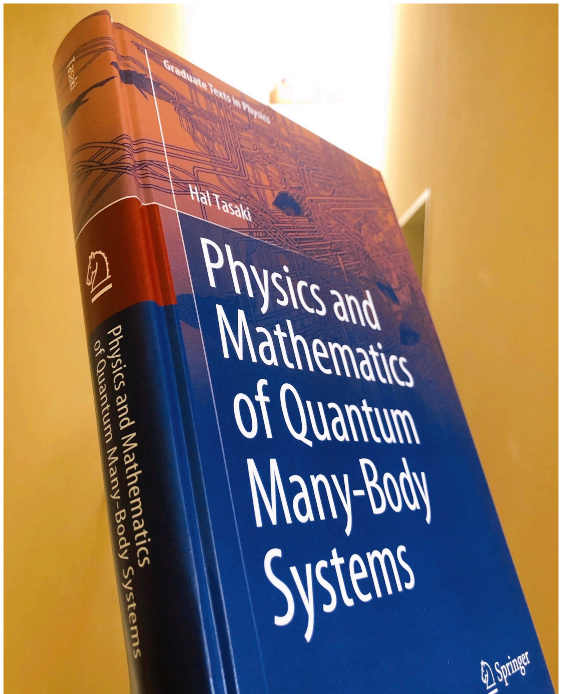
Franz Wegner (1940-)

Vadim Berezinskii (1935-1980)

The absence of ferromagnetic order in the two-dimensional XY model

part 3 Wegner's harmonic approximation

Advanced Topics in Statistical Physics
by Hal Tasaki



Tasaki (2020)

general fact. Under the same conditions, it is proved that the truncated correlation function of any local operators $\hat{A}$ and $\hat{B}$ exhibits exponential decay as

$$\left|\langle \hat{A}\,\tau_x(\hat{B})\rangle^{\infty}_{\beta,h} - \langle \hat{A}\rangle^{\infty}_{\beta,h}\langle \hat{B}\rangle^{\infty}_{\beta,h}\right| \le (\text{constant})\,\exp\Big[-\frac{|x|}{\xi(\beta,h)}\Big], \tag{4.4.9}$$

with $\xi(\beta, h) \in (0, \infty)$, where τ_x is the translation map defined in (4.3.3). Then (4.4.6) and (4.4.8) are special cases.

These results about disordered equilibrium states are proved for a much larger class of models than the Heisenberg model. Basically any model with translationally invariant short ranged interactions can be treated in one dimension. See [4]. At sufficiently high temperature (in any dimensions), these results can be proved for any model with (not necessarily translation invariant) short ranged interactions. The proof makes use of the technique of cluster expansion. See, e.g., [21, 50, 61].

4.4.2 *Berezinskii's Harmonic Approximation*

The nature of LRO and SSB in equilibrium states also depends crucially on the dimensionality of the system. To see the essence of the dependence, we shall review an approximate theory due to Berezinskii [8] about the correlation function of the classical XY model. Although the theory is simple, it sheds light on essential properties of spin systems with continuous symmetry. In fact the theory was a motivation for the rigorous method of McBryan and Spencer [44], which we shall discuss in Sect. 4.4.3. We also note that it is widely believed that the basic nature of equilibrium states is common for classical and quantum spin systems.

We consider the classical XY model, in which each lattice site $x \in \Lambda_L$ is associated with a classical XY spin, i.e., a two-dimensional vector $\vec{S}_x = (S^{(1)}_x, S^{(2)}_x) \in \mathbb{R}^2$ such that $|\vec{S}_x| = 1$. The Hamiltonian of the ferromagnetic model without external magnetic filed is

$$H^{\mathrm{XY}} = \sum_{\{x,y\}\in\mathscr{B}_L} (1 - \vec{S}_x \cdot \vec{S}_y), \tag{4.4.10}$$

where the constant 1 is inserted for later convenience. It is useful to represent the spin by the angle variable $\theta_x \in [-\pi, \pi)$ as $\vec{S}_x = (\cos\theta_x, \sin\theta_x)$. Then we see that $\vec{S}_x \cdot \vec{S}_y = \cos(\theta_x - \theta_y)$. The Hamiltonian H^{XY} is minimized when $\theta_x - \theta_y = 0$ for any $\{x, y\} \in \mathscr{B}_L$. Thus all the spins point in the same direction in any ground state.[42]

The thermal expectation value of an arbitrary function f of spins is defined as

[42] A ground state of a classical spin system is a spin configuration that minimi

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§ Gaussian approximation of the XY model → not rigorous

1

the ground states (the minimum energy states)

$h=0$ $\quad \theta_u = \theta$ for all $u \in \Lambda_L$ with arbitrary θ

$h>0$ $\quad \theta_u = 0$ for all $u \in \Lambda_L$

$\beta \gg 1$ → configurations are mostly close to the ground states

$h=0$ $\quad |\theta_u - \theta_v| \ll 1$ for $\{u,v\} \in \mathcal{B}_L$

$h>0$ $\quad |\theta_u| \ll 1$ for all $u \in \Lambda_L$ (and $|\theta_u - \theta_v| \ll 1$)

($|S|$: the number of elements in a finite set S)

$$H_{L,h}(\Theta) = -|\mathcal{B}_L| - h|\Lambda_L| - \sum_{\{u,v\}\in\mathcal{B}_L}{}' (\cos(\theta_u - \theta_v) - 1) - h \sum_{u\in\Lambda_L}{}' (\cos\theta_u - 1)$$

$$\simeq -|\mathcal{B}_L| - h|\Lambda_L| + \underbrace{\frac{1}{2}\sum_{\{u,v\}\in\mathcal{B}_L}{}' (\theta_u - \theta_v)^2 + \frac{h}{2}\sum_{u\in\Lambda_L}{}' (\theta_u)^2}_{H^G_{L,h}(\Theta)} \qquad (1)$$

bold approximations

$\theta_u \in [-\pi, \pi) \longrightarrow \theta_u \in \mathbb{R} \qquad (2)$

§ Gaussian model and the basic identity for the correlation function

Gaussian model

$$\theta_u \in \mathbb{R} \quad (u \in \Lambda_L) \qquad \Theta = (\theta_u)_{u \in \Lambda_L}$$

Hamiltonian

$$H^G_{L,h}(\Theta) = \frac{1}{2} \sum_{\{u,v\} \in \mathcal{B}_L} (\theta_u - \theta_v)^2 + \frac{h}{2} \sum_{u \in \Lambda_L} (\theta_u)^2 \qquad (1)$$

partition function

$$Z^G_L(\beta, h) = \int_{-\infty}^{\infty} d\Theta \, e^{-\beta H^G_{L,h}(\Theta)} \qquad (2)$$

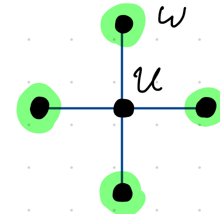
equilibrium expectation

$$\langle \cdots \rangle^G_{L,\beta,h} = \frac{1}{Z^G_L(\beta,h)} \int_{-\infty}^{\infty} d\Theta \, (\cdots) \, e^{-\beta H^G_{L,h}(\Theta)} \qquad (3)$$

$$\text{with} \quad \int_{-\infty}^{\infty} d\Theta = \prod_{u \in \Lambda_L} \int_{-\infty}^{\infty} d\theta_u \qquad (4)$$

identity for the correlation function

$$\frac{\partial}{\partial\Theta_u} H^{G}_{L,h}(\Theta) = \sum_{w\in N(u)} (\Theta_u - \Theta_w) + h\,\Theta_u \qquad (1)$$



$$N(u) = \{ w \mid \{u,w\} \in B_L \} \qquad (2)$$

$$\left\langle \Theta_v \Big\{ \sum_{w\in N(u)} (\Theta_u - \Theta_w) + h\,\Theta_u \Big\} \right\rangle^{G}_{L,\beta,h} = \frac{1}{Z^{G}_{L}(\beta,h)} \int_{-\infty}^{\infty} d\Theta\, \Theta_v \left(\frac{\partial}{\partial\Theta_u} H^{G}_{L,h}(\Theta) \right) e^{-\beta H^{G}_{L,h}(\Theta)}$$

$$= -\frac{1}{\beta}\frac{1}{Z^{G}_{L}(\beta,h)} \int_{-\infty}^{\infty} d\Theta\, \Theta_v \frac{\partial}{\partial\Theta_u} e^{-\beta H^{G}_{L,h}(\Theta)} = \frac{1}{\beta}\frac{1}{Z^{G}_{L}(\beta,h)} \int_{-\infty}^{\infty} d\Theta\, \boxed{\frac{\partial\Theta_v}{\partial\Theta_u}}\, e^{-\beta H^{G}_{L,h}(\Theta)}$$

$$= \delta_{uv}$$

$$= \frac{1}{\beta}\delta_{u,v} \qquad (3)$$

$$\sum_{w\in N(u)} \left(g^{(v)}_u - g^{(v)}_w \right) + h\, g^{(v)}_u = \frac{1}{\beta}\delta_{uv} \qquad (4)$$

$$g^{(v)}_u = \langle \Theta_v \Theta_u \rangle^{G}_{L,\beta,h} \qquad (5)$$

regard v as fixed

4

$$g_u^{(v)} = \langle \theta_v \theta_u \rangle^G_{L,\beta,h} \quad (1) \qquad \sum_{w \in \mathcal{N}(u)} \left(g_u^{(v)} - g_w^{(v)} \right) + h\, g_u^{(v)} = \frac{1}{\beta} \delta_{u,v} \quad (2)$$

lattice Laplacian $\Delta = (\Delta_{uw})_{u,w \in \Lambda_L}$ (3)

$$\boxed{\Delta_{uw} = \begin{cases} -2d & u = w \\ 1 & \{u,w\} \in \mathcal{B}_L \\ 0 & \text{otherwise} \end{cases}} \quad (4)$$

$$\sum_{w \in \mathcal{N}(u)} \left(g_u^{(v)} - g_w^{(v)} \right) = -\sum_{w \in \Lambda_L} \Delta_{uw}\, g_w^{(v)} \quad (5)$$

$$(2) \Leftrightarrow \boxed{\sum_{w \in \Lambda_L} \underbrace{\left(-\Delta_{uw} + h\, \delta_{uw} \right)}_{(-\Delta + hI)_{uw}} g_w^{(v)} = \frac{1}{\beta} \delta_{uv}} \quad (6)$$

the eigenvalues of $-\Delta$ are nonegative. p.5

$(-\Delta + hI)$ is invertible for $h > 0$

(2) has a unique solution

$$g_u^{(v)} = \frac{1}{\beta} \sum_{w \in \Lambda_L} \left((-\Delta + hI)^{-1} \right)_{uw} \delta_{wv} = \frac{1}{\beta} \left((-\Delta + hI)^{-1} \right)_{uv} \quad (7)$$

III eigenvalues and eigenvectors of Δ

$$\Delta_{uw} = \begin{cases} -2d & u=w \\ 1 & \{u,w\}\in B_L \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

Δ real symmetric $\rightarrow$ eigenvectors form an orthonormal basis

for any $f_u \in \mathbb{C}$

$$\sum_{u,w\in\Lambda_L} (f_u)^* \Delta_{uw} f_v = -\sum_{\{u,w\}\in B_L} |f_u - f_w|^2 \qquad (2)$$

$$\left(\text{RHS} = \sum_{\{u,w\}\in B_L} \{-|f_u|^2 + f_u^* f_w - |f_w|^2 + f_u f_w^*\} = \sum_{u\in\Lambda_L} \{-2d|f_u|^2 + \sum_{w\in N(u)} f_u^* f_w\} = \text{LHS} \right)$$

▲ (2) $\rightarrow$ $\sum_{u,w} (f_u)^* \Delta_{uw} f_w \le 0$ $\rightarrow$ all the eigenvalues of Δ are nonpositive

▲ $\sum_w \Delta_{uw} = 0$ $\rightarrow$ $(1,1,\ldots,1)^t$ is the eigenvector for eigenvalue 0.

▲ (2) $\rightarrow$ $\sum_{u,w} (f_u)^* \Delta_{uw} f_w = 0$ only when $f_u = \text{const.}$ for all $u \in \Lambda_L$

$\rightarrow$ The eigenvalue 0 is nondegenerate (all other eigenvalues < 0)

§ Spontaneous magnetization

$$m_s^G(\beta) = \lim_{h\downarrow 0}\lim_{L\uparrow\infty} \left\langle \frac{1}{L^d}\sum_{u\in\Lambda_L} S_u^{(x)} \right\rangle_{L,\beta,h}^G = \lim_{h\downarrow 0}\lim_{L\uparrow\infty} \langle S_v^{(x)} \rangle_{L,\beta,h}^G \quad (1)$$

part 2 - p3 - (1)

property of the Gaussian integral (p.15)

$$\langle S_v^{(x)} \rangle_{L,\beta,h}^G = \langle e^{i\theta_v} \rangle_{L,\beta,h}^G = \exp\left[-\frac{1}{2}\langle (\theta_v)^2 \rangle_{L,\beta,h}^G\right] = e^{-\frac{1}{2}\boxed{g_v^{(v)}}} \quad (2)$$

we shall solve P4 - (2) = (6) (unique solution exists if $h>0$)

fix v $\quad g_u^{(v)} \to g_u$

$$\sum_{w\in\Lambda_L} (-\Delta_{uw} + h\,\delta_{uw})\, g_w = \frac{1}{\beta}\delta_{uv} \quad (3)$$

$$\sum_{j=1}^{d} (2g_u - g_{u+e_j} - g_{u-e_j}) + h\, g_u = \frac{1}{\beta}\delta_{uv} \quad (4)$$

jth component

$e_j = (0, \dots, 0, 1, 0, \dots, 0)$

Fourier transformation

$$K_L = \left\{ (k_1, \ldots, k_d) \;\middle|\; k_j = \frac{2\pi}{L} n_j ,\; n_j = 0, \pm 1, \ldots, \pm \frac{L-1}{2} \right\} \quad (1)$$

$$L^{-d} \sum_{u \in \Lambda_L} e^{ik\cdot u} = \prod_{j=1}^{d} \delta_{k_j,0} \quad (2) \qquad L^{-d} \sum_{k \in K_L} e^{ik\cdot u} = \prod_{j=1}^{d} \delta_{u_j,0} \quad (3) \qquad \text{with } k\cdot u = \sum_{j=1}^{d} k_j u_j \quad (4)$$

$$\tilde{g}_k = L^{-d/2} \sum_{u \in \Lambda_L} e^{-ik\cdot u} g_u \qquad (k \in K_L) \qquad (5)$$

$$g_u = L^{-d/2} \sum_{k \in K_L} e^{ik\cdot u} \tilde{g}_k \qquad (u \in \Lambda_L) \qquad (6)$$

p6-(4) $$\sum_{j=1}^{d} \left(2 g_u - g_{u+e_j} - g_{u-e_j} \right) + h\, g_u = \frac{1}{\beta} \delta_{uv} \qquad (7)$$

$\Downarrow$ substitute (6)

$$L^{-d/2} \sum_{k' \in K_L} \Big\{ \underbrace{\sum_{j=1}^{d} \left(2 - e^{ik'_j} - e^{-ik'_j} \right)}_{= \,\epsilon(k')} + h \Big\} e^{ik'\cdot u}\, \tilde{g}_{k'} = \frac{1}{\beta} \delta_{uv} \qquad (8)$$

$|k| \ll 1$

$$\epsilon(k) = 2 \sum_{j=1}^{d} (1 - \cos k_j) \simeq |k|^2 \qquad (9)$$

$$L^{-d/2} \sum_{k' \in K_L} \{\epsilon(k') + h\} e^{ik' \cdot u} \, \tilde{g}_{k'} = \frac{1}{\beta} \delta_{uv} \quad (1)$$

$$\downarrow \quad L^{-d/2} \sum_{u \in \Lambda_L} e^{-ik \cdot u} (\mathrm{LHS}) = L^{-d/2} \sum_{u \in \Lambda_L} e^{-ik \cdot u} (\mathrm{RHS}) \quad (2)$$

$$\{\epsilon(k) + h\} \tilde{g}_k = L^{-\frac{d}{2}} \frac{1}{\beta} e^{-ik \cdot v} \quad (3) \quad \longrightarrow \quad \tilde{g}_k = L^{-\frac{d}{2}} \frac{1}{\beta} \frac{1}{\epsilon(k) + h} \quad (4)$$

p7-(6)

inverse Fourier transformation

$$g_u^{(v)} = \frac{1}{\beta} \frac{1}{L^d} \sum_{k \in K_L} \frac{e^{ik \cdot (u-v)}}{\epsilon(k) + h} \quad (5)$$

$$\left(\frac{2\pi}{L}\right)^d \sum_{k \in K_L} (\cdots) \xrightarrow[L \uparrow \infty]{} \prod_{j=1}^{d} \int_{-\pi}^{\pi} dk_j \, (\cdots) = \int_{[-\pi,\pi]^d} dk \, (\cdots) \quad (6)$$

we thus get

$$\lim_{L \uparrow \infty} g_u^{(v)} = \frac{1}{\beta (2\pi)^d} \underbrace{\int_{[-\pi,\pi]^d} dk \, \frac{e^{ik \cdot (u-v)}}{\epsilon(k) + h}}_{= I_d(h)} \quad (7)$$

9

p6-(1),(2)

$$m_s^G(\beta) = \lim_{h\downarrow 0}\lim_{L\uparrow\infty} e^{-\frac{1}{2}g_u^{(n)}} = \exp\left[-\frac{1}{2}\frac{1}{(2\pi)^d}\frac{1}{\beta}\boxed{\lim_{h\downarrow 0} I_d(h)}\right] \quad (1)$$

$$I_d(h) = \int_{[-\pi,\pi]^d} dk \frac{1}{\epsilon(k)+h} \quad (2)$$

$$I_d(0) = \int_{[-\pi,\pi]^d} dk \frac{1}{\epsilon(k)} \sim \int_{|k|\le\pi} dk \frac{1}{|k|^2} \overset{p=|k|}{\sim} \int_0^{\pi} dp \frac{p^{d-1}}{p^2} \begin{cases} <\infty & d\geq 3 \\ =\infty & d=1,2 \end{cases} \quad (3)$$

$d\geq 3$ $\quad m_s^G(\beta)\simeq 1$ for large β $\quad$ ferromagnetic order is stable

$d=1,2$ $\quad$ the approximation fails because of the infrared divergence

ferromagnetic order is unstable even at large β

(Hohenberg-Mermin-Wagner theorem)

§ correlation function at $h=0$

property of the Gaussian integral (p.15)

part 2-p3-(2)

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,h} = \langle e^{i(\theta_u-\theta_v)} \rangle^G_{L,\beta,h} = e^{-\frac{1}{2}\langle(\theta_u-\theta_v)^2\rangle^G_{L,\beta,h}} \quad (1)$$

evaluate this for $h=0$

regard u, v as fixed

$$\langle(\theta_u-\theta_v)^2\rangle^G_{L,\beta,0} = \Phi^{(u,v)}_u - \Phi^{(u,v)}_v \quad (2)$$

with $h=0$

$$\text{with} \quad \Phi^{(u,v)}_w = \langle \theta_w(\theta_u-\theta_v)\rangle^G_{L,\beta,0} = g^{(u)}_w - g^{(v)}_w \quad (3)$$

p4-(6)
$$-\sum_{w'\in\Lambda_L} \Delta_{ww'}\Phi^{(u,v)}_{w'} = \frac{1}{\beta}(\delta_{uw}-\delta_{vw}) \quad (4)$$

p5 → Δ has non-degenerate eigenvalue 0 with unique eigenvector $(1,1,\cdots,1)^t$

(4) has a unique solution because $\delta_{uw'}-\delta_{vw'}$ is orthogonal to $(1,1,\cdots,1)^t$

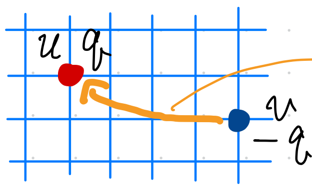
$$\boxed{\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,0} = e^{-\frac{1}{2}\{\Phi^{(u,v)}_u - \Phi^{(u,v)}_v\}}} \quad (5)$$

▷ analogy with electrostatics → one may use k-space integrals

$$-\sum_{w' \in \Lambda_L} \Delta_{ww'} \Phi^{(u,v)}_{w'} = \frac{1}{\beta}(\delta_{uw} - \delta_{vw}) \qquad (1)$$

discretized Poisson's equation for electric potential $\Phi^{(u,v)}_{w}$

charge q at u charge $-q$ at v $q = \frac{1}{\beta}$

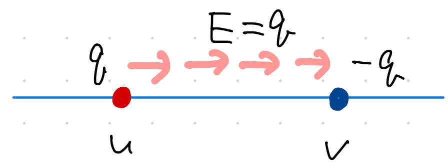


potential difference

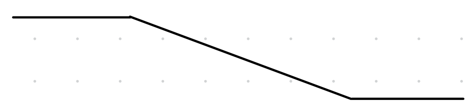
$$\Phi^{(u,v)}_{u} - \Phi^{(u,v)}_{v} > 0 \qquad (2)$$

$d=1$

electric field

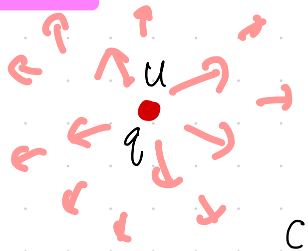


electric potential $\Phi^{(u,v)}_{w}$



$$\Phi^{(u,v)}_{u} - \Phi^{(u,v)}_{v} = q\,|u-v| = \frac{1}{\beta}|u-v| \qquad (3)$$

$d=2$ electric field generated by a charge at u (in the infinite lattice)

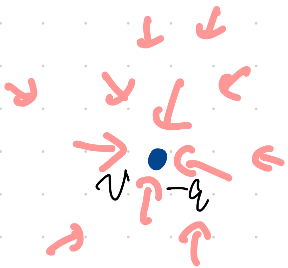


$$|\mathbb{E}(w)| \simeq \frac{q}{2\pi|w-u|} \quad \text{for } |w-u| \gg 1 \qquad (1)$$

no divergence!

corresponding electric potential

$$\varphi_w^{(u,q)} \begin{cases} = C_2 q & w=u \\ \simeq -\dfrac{q}{2\pi}\log|w-u| & |w-u| \gg 1 \end{cases} \qquad (2)$$



similarly

$$\varphi_w^{(v,-q)} \begin{cases} = -C_2 q & w=u \\ \simeq \dfrac{q}{2\pi}\log|w-v| & |w-v| \gg 1 \end{cases} \qquad (3)$$

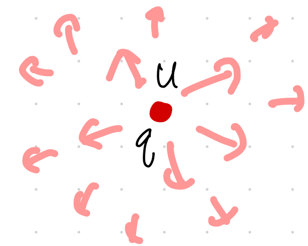
$$\Phi_w^{(u,v)} = \varphi_w^{(u,q)} + \varphi_w^{(v,-q)} \qquad (4)$$

$$\Phi_u^{(u,v)} - \Phi_v^{(u,v)} \simeq 2C_2 q + \frac{q}{\pi}\log|u-v|$$

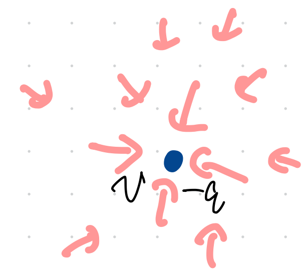
$$= 2\frac{C_2}{\beta} + \frac{1}{\pi\beta}\log|u-v| \qquad (5)$$

$d=3$ the familar case

$$\varphi_w^{(u,q)} \begin{cases} = C_3\, q & w=u \\ \simeq \dfrac{q}{4\pi |w-u|} & |w-u| \gg 1 \end{cases} \quad (1)$$



$$\varphi_w^{(v,-q)} \begin{cases} = -C_3\, q & w=u \\ \simeq -\dfrac{q}{4\pi |w-v|} & |w-u| \gg 1 \end{cases} \quad (2)$$



$$\Phi_w^{(u,v)} = \varphi_w^{(u,q)} + \varphi_w^{(v,-q)} \quad (3)$$

$$\Phi_u^{(u,v)} - \Phi_v^{(u,v)} \simeq 2 C_3\, q - \frac{q}{2\pi |u-v|}$$

$$= 2\frac{C_3}{\beta} - \frac{1}{2\pi\beta |u-v|} \quad (4)$$

iii behavior of the correlation function Wegner (1967)

p10-(5) $\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,0} = e^{-\frac{1}{2}\{\Phi_u^{(u,v)} - \Phi_v^{(u,v)}\}}$

for $1 \ll |u-v| \lesssim \frac{L}{2}$

$d=1$ $\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,0} \sim e^{-\frac{1}{2\beta}|u-v|}$ (1)

exponential decay

$d=2$ $\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,0} \sim e^{-\frac{C_2}{\beta} - \frac{1}{2\pi\beta}\log|u-v|} = e^{-\frac{C_2}{\beta}} |u-v|^{-\frac{1}{2\pi\beta}}$ (2)

power law decay

$d=3$ $\langle \vec{S}_u \cdot \vec{S}_v \rangle^G_{L,\beta,0} \sim e^{-\frac{C_3}{\beta} + \frac{1}{4\pi\beta|u-v|}} \simeq e^{-\frac{C_3}{\beta}} > 0$ (3)

long-range order (LRO)

§ Gaussian integral

Λ a finite set

$(A_{uv})_{u,v\in\Lambda}$ real symmetric matrix with positive eigenvalues

we don't use this

$$Z = \int_{-\infty}^{\infty} d\Theta \, \exp\left[-\frac{1}{2}\sum_{u,v\in\Lambda} \theta_u A_{uv} \theta_v\right] = \boxed{\frac{(2\pi)^{|\Lambda|/2}}{\sqrt{\det A}}} \qquad (1)$$

$$\langle \cdots \rangle = Z^{-1} \int_{-\infty}^{\infty} d\Theta \, (\cdots) \exp\left[-\frac{1}{2}\sum_{u,v\in\Lambda} \theta_u A_{uv} \theta_v\right] \qquad (2)$$

for any $a_u \in \mathbb{C}$, $\left\langle \exp\left[\sum_{u\in\Lambda} a_u \theta_u\right]\right\rangle = \exp\left[\frac{1}{2}\sum_{u,v\in\Lambda} a_u \langle \theta_u \theta_v\rangle a_v\right]$ (3)

P6-(2) $a_v = i$, $a_w = 0$ $(w \neq v)$ $\quad \langle e^{i\theta_v}\rangle = e^{\frac{1}{2} i^2 \langle \theta_v^2\rangle} = e^{-\frac{1}{2}\langle \theta_v^2\rangle}$ (4)

P10-(1) $a_u = i$, $a_v = -i$, $a_w = 0$ $(w \neq u, v)$

$$\langle e^{i(\theta_u - \theta_v)}\rangle = e^{\frac{1}{2}\{i^2\langle \theta_u^2\rangle + (-i)^2\langle \theta_v^2\rangle + 2\langle \theta_u \theta_v\rangle\}} = e^{-\frac{1}{2}\langle (\theta_u - \theta_v)^2\rangle} \qquad (5)$$

derivation of P15-(3)

we shall show
$$\left\langle \exp\left[\sum_{u\in\Lambda} a_u \theta_u\right]\right\rangle = \exp\left[\frac{1}{2}\sum_{u,v\in\Lambda} a_u (A^{-1})_{uv} a_v\right] \quad (1)$$

→ P15-(3)

then
$$\langle \theta_u \theta_v \rangle = \frac{\partial^2}{\partial a_u \partial a_v} \left\langle \exp\left[\sum_w a_w \theta_w\right]\right\rangle \Big|_{a_w=0}$$
$$= \frac{\partial^2}{\partial a_u \partial a_v} \exp\left[\frac{1}{2}\sum_{w,w'} a_w (A^{-1})_{ww'} a_{w'}\right] \Big|_{a_w=0} = (A^{-1})_{uv} \quad (2)$$

for any $b_u \in \mathbb{C}$

complex Gaussian integral
$$\int_{-\infty}^{\infty} d\Theta \exp\left[-\frac{1}{2}\sum_{u,v\in\Lambda} (\theta_u - b_u) A_{uv} (\theta_v - b_v)\right] = Z \quad (3)$$

chose
$$b_u = \sum_v (A^{-1})_{uv} a_v = \sum_v a_v (A^{-1})_{vu} \quad (4)$$

$$\sum_{u,v} (\theta_u - b_u) A_{uv} (\theta_v - b_v) = \sum_{u,v} (\theta_u A_{uv} \theta_v - b_u A_{uv} \theta_v - \theta_u A_{uv} b_v + b_u A_{uv} b_v)$$
$$= \sum_{u,v} \theta_u A_{uv} \theta_v - 2\sum_u a_u \theta_u + \sum_{u,v} a_u (A^{-1})_{uv} a_v \quad (5)$$

$$(\text{LHS of } (3)) = \int_{-\infty}^{\infty} d\Theta \, e^{\sum_u a_u \theta_u} e^{-\frac{1}{2}\sum_{u,v} \theta_u A_{uv} \theta_v} e^{-\frac{1}{2}\sum_{u,v} a_u (A^{-1})_{uv} a_v} \quad (6)$$

appendix useful exercises on Wick's theorem

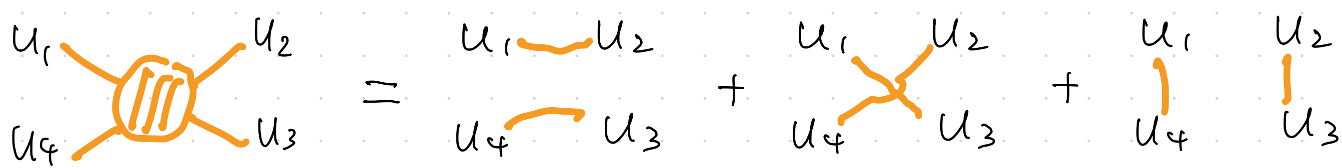
$F(\Theta)$ any function of Θ

$$\langle \theta_u F(\Theta) \rangle = \sum_{w \in \Omega} \langle \theta_u \theta_w \rangle \left\langle \frac{\partial F(\Theta)}{\partial \theta_w} \right\rangle \quad (1)$$

proof: integration by parts as in p. 3. with p16-(2)

applications → Wick's theorem

$$\langle \theta_{u_1} \theta_{u_2} \theta_{u_3} \theta_{u_4} \rangle = \langle \theta_{u_1} \theta_{u_2} \rangle \langle \theta_{u_3} \theta_{u_4} \rangle + \langle \theta_{u_1} \theta_{u_3} \rangle \langle \theta_{u_2} \theta_{u_4} \rangle + \langle \theta_{u_1} \theta_{u_4} \rangle \langle \theta_{u_2} \theta_{u_3} \rangle \quad (2)$$



$$\langle \theta_{u_1} \cdots \theta_{u_{2n}} \rangle = \sum_{\substack{\text{partitions} \\ \text{into distinct} \\ \text{pairs}}} \langle \theta_{p_1} \theta_{q_1} \rangle \cdots \langle \theta_{p_n} \theta_{q_n} \rangle \quad (3)$$

partitions into distinct pairs → $(2n-1)!!$ terms