The absence of ferromagnetic order in the two-dimensional XY model

part 2 the XY model: the definition and the representation in terms of phases

Advanced Topics in Statistical Physics
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§ definitions

lattice $\Lambda_L = \{1, 2, \dots, L\}^d$ (1) **sites** $u, v, w, \dots \in \Lambda_L$

set of bonds $\mathcal{B}_L = \{\{u, v\} \mid u$ and v are nearest neighbors (periodic b.c.)$\}$

spin $\vec{S}_u = (S_u^{(x)}, S_u^{(y)}) \in \mathbb{R}^2$ s.t. $|\vec{S}_u| = 1$ $\qquad \mathbb{S} = (\vec{S}_u)_{u \in \Lambda_L}$

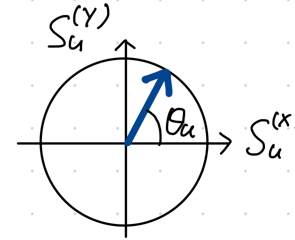
Hamiltonian $$H_{L,h}(\mathbb{S}) = -\sum_{\{u,v\} \in \mathcal{B}_L} \vec{S}_u \cdot \vec{S}_v - h \sum_{u \in \Lambda_L} S_u^{(x)} \qquad (2)$$

partition function
$$Z_L(\beta, h) = \int d\mathbb{S}\; e^{-\beta H_{L,h}(\mathbb{S})} \qquad (3) \qquad \beta = \frac{1}{k_B T} \in (0, \infty)$$

equilibrium expectation
$$\langle \dots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta, h)} \int d\mathbb{S}\, (\dots)\, e^{-\beta H_{L,h}(\mathbb{S})} \qquad (4)$$

with $$\int d\mathbb{S}\,(\dots) = \prod_{u \in \Lambda_L} \iint_{|\vec{S}_u| = 1} dS_u^{(x)}\, dS_u^{(y)}\, (\dots) \qquad (5)$$

§ representation in terms of phases



$$\vec{S}_u = (\cos\theta_u, \sin\theta_u) \qquad \theta_u \in [0, 2\pi)$$

$$\vec{S}_u \cdot \vec{S}_v = \cos(\theta_u - \theta_v) \qquad (1)$$

$$\Theta = (\theta_u)_{u \in \Lambda_L}$$

$$\hat{H}_{L,h}(\Theta) = -\sum_{\{u,v\} \in \mathcal{B}_L} \cos(\theta_u - \theta_v) - h \sum_{u \in \Lambda_L} \cos\theta_u \qquad (2)$$

$$Z_L(\beta, h) = \int d\Theta \, e^{-\beta H_{L,h}(\Theta)} \qquad (3)$$

$$\langle \cdots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta,h)} \int d\Theta \, (\cdots) \, e^{-\beta H_{L,h}(\Theta)} \qquad (4)$$

$$\int d\Theta \, (\cdots) = \prod_{u \in \Lambda_L} \int_0^{2\pi} d\theta_u \, (\cdots) \left(= \int d\mathbf{S} \, (\cdots) \right) \qquad (5)$$

Symmetry

$H_{L,h}(\Theta)$ invariant under $\theta_u \to -\theta_u$ for all $u \in \Lambda_L$

$(S_u^{(x)}, S_u^{(y)}) \to (S_u^{(x)}, -S_u^{(y)})$

We thus have

$$\langle S_u^{(x)} \rangle_{L,\beta,h} = \langle \cos\theta_u \rangle_{L,\beta,h} = \langle e^{i\theta_u} \rangle_{L,\beta,h} \quad (1)$$

$\langle \sin\theta_u \rangle_{L,\beta,h} = 0$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,h} = \langle \cos(\theta_u - \theta_v) \rangle_{L,\beta,h} = \langle e^{i(\theta_u - \theta_v)} \rangle_{L,\beta,h} \quad (2)$$

$\langle \sin(\theta_u - \theta_v) \rangle_{L,\beta,h} = 0$