The absence of ferromagnetic order in the two-dimensional XY model

part 1 background and main results

Advanced Topics in Statistical Physics
by Hal Tasaki

§ three standard classical spin systems

$d=1$ chain
$d=2$ square lattice
$d=3$ cubic lattice

lattice and the set of bonds

dimension $d = 1, 2, 3, \ldots$

d-dimensional hyper cubic lattice $\Lambda_L = \{1, 2, \ldots, L\}^d$ (1)

sites $u, v, w, \cdots \in \Lambda_L$ $u = (u_1, u_2, \ldots, u_d)$ with $u_j \in \{1, \cdots, L\}$

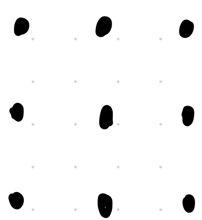
set of bonds $\mathcal{B}_L = \{\{u, v\} \mid u$ and v are nearest neighbors$\}$ (2)

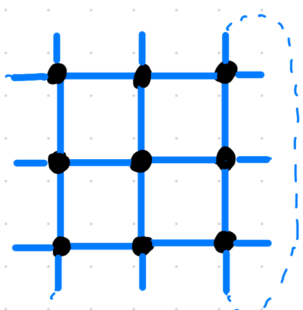
bond $\{u, v\} = \{v, u\} \in \mathcal{B}_L$

$|u - v| = 1$

OR

$$\begin{cases} u = (u_1, \cdots, L, \cdots, u_d) \\ v = (u_1, \cdots, 1, \cdots, u_d) \end{cases}$$

Λ_3 

$\mathcal{B}_3$ 

we only use
periodic boundary conditions

(iii) ferromagnetic Heisenberg model

spin at site $u \in \Lambda_L$ $\quad \vec{S}_u = (S_u^{(x)}, S_u^{(y)}, S_u^{(z)}) \in \mathbb{R}^3$ s.t. $|\vec{S}_u| = 1$

spin configuration $\quad \mathbb{S} = (\vec{S}_u)_{u \in \Lambda_L}$

Heisenberg interaction

magnetic field

Hamiltonian

$$H_{L,h}(\mathbb{S}) = -\sum_{\{u,v\}\in \mathcal{B}_L} \boxed{\vec{S}_u \cdot \vec{S}_v} - \boxed{h} \sum_{u \in \Lambda_L} S_u^{(x)} \qquad (1)$$

partition function

$$Z_L(\beta, h) = \int d\mathbb{S}\; e^{-\beta H_{L,h}(\mathbb{S})} \qquad (2) \qquad \beta = \frac{1}{k_B T}$$

$$\int d\mathbb{S}\,(\cdots) = \prod_{u \in \Lambda_L} \iiint_{|\vec{S}_u| = 1} dS_u^{(x)}\, dS_u^{(y)}\, dS_u^{(z)}\,(\cdots) \qquad (3)$$

expectation value in the equlibrium state

$$\langle \cdots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta, h)} \int d\mathbb{S}\,(\cdots)\, e^{-\beta H_{L,h}(\mathbb{S})} \qquad (4)$$

(iii) ferromagnetic XY model

spin at site $u \in \Lambda_L$ $\quad \vec{S}_u = (S_u^{(x)}, S_u^{(y)}) \in \mathbb{R}^2$ s.t. $|\vec{S}_u| = 1$

spin configuration $\quad \mathbb{S} = (\vec{S}_u)_{u \in \Lambda_L}$

Hamiltonian $\quad H_{L,h}(\mathbb{S}) = -\sum_{\{u,v\} \in \mathcal{B}_L} \vec{S}_u \cdot \vec{S}_v - h \sum_{u \in \Lambda_L} S_u^{(x)}$ (1)

partition function $\quad Z_L(\beta, h) = \int d\mathbb{S}\, e^{-\beta H_{L,h}(\mathbb{S})}$ (2)

with $\displaystyle \int d\mathbb{S}\, (\cdots) = \prod_{u \in \Lambda_L} \iint_{|\vec{S}_u| = 1} dS_u^{(x)} dS_u^{(y)} (\cdots)$ (3)

expectation value in the equlibrium state

$$\langle \cdots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta,h)} \int d\mathbb{S}\, (\cdots)\, e^{-\beta H_{L,h}(\mathbb{S})} \quad (4)$$

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▷ ferromagnetic Ising model (Lenz-Ising model)

spin at site $u \in \Lambda_L$ $\sigma_u \in \{1, -1\}$

spin configuration $\boldsymbol{\sigma} = (\sigma_u)_{u \in \Lambda_L}$

Hamiltonian $$H_{L,h}(\boldsymbol{\sigma}) = -\sum_{\{u,v\} \in \mathcal{B}_L} \sigma_u \sigma_v - h \sum_{u \in \Lambda_L} \sigma_u \qquad (1)$$

partition function $$Z_L(\beta, h) = \sum_{\boldsymbol{\sigma}}{}' e^{-\beta H_{L,h}(\boldsymbol{\sigma})} \qquad (2)$$

with $$\sum_{\boldsymbol{\sigma}}{}'(\cdots) = \prod_{u \in \Lambda_L} \sum_{\sigma_u = \pm 1} (\cdots) \qquad (3)$$

expectation value in the equlibrium state

$$\langle \cdots \rangle_{L,\beta,h} = \frac{1}{Z_L(\beta,h)} \sum_{\boldsymbol{\sigma}}{}' (\cdots) e^{-\beta H_{L,h}(\boldsymbol{\sigma})} \qquad (4)$$

iii) characteristics of the three models

Heisenberg model

(antiferromagnets are more realistic)

natural and realistic model of interacting spins

Ising model

model of ferromagnets with strong axial anisotropy

the simplest model for studying collective behavior of interacting spins

XY model

model of ferromagnets with strong planar anisotropy

simple theoretical model with continuous symmetry

effective model of superconductors and superfluids

(U(1) order parameter ⟷ XY spin)

iii) symmetry of the models with $h=0$

Ising model $\quad H_{L,0}(\boldsymbol{\sigma}) = -\sum_{\{u,v\}\in B_L} \sigma_u \sigma_v \quad (1)$

invariant under global spinflip $\quad H_{L,0}(\boldsymbol{\sigma}) = H_{L,0}(-\boldsymbol{\sigma}) \quad (2)$

where $\quad -\boldsymbol{\sigma} = (-\sigma_u)_{u\in\Lambda_L} \quad (3)$

the model has discrete $\mathbb{Z}_2$ symmetry

XY and Heisenberg model $\quad H_{L,0}(\boldsymbol{S}) = -\sum_{\{u,v\}\in B_L} \vec{S}_u \cdot \vec{S}_v \quad (4)$

invariant under global spin rotation $\quad H_{L,0}(\boldsymbol{S}) = H_{L,0}(R\boldsymbol{S}) \quad (5)$

where $\quad R\boldsymbol{S} = (R\vec{S}_u)_{u\in\Lambda_L} \quad (6) \quad$ with $R \in O(n)$

$n\times n$ orthogonal matrix

$n=2$ XY
$n=3$ Heisenberg

the model has continuous $O(n)$ symmetry

§ ferromagnetic order in the Ising model : SSB and LRO

$d \geq 2$ the model undergoes a phase transition at β_c $(0 < \beta_c < \infty)$

ferromagnetic order at $\beta > \beta_c$

▷ spontaneous symmetry breaking (SSB)

(translational invariance)

$$\boxed{m_s(\beta)} = \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \left\langle \frac{1}{L^d} \sum_{u \in \Lambda_L} \sigma_u \right\rangle_{L,\beta,h} = \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \langle \sigma_u \rangle_{L,\beta,h} \begin{cases} = 0, & \beta \leq \beta_c \\ > 0, & \beta > \beta_c \end{cases} \quad (1)$$

spontaneous magnetization

magnetization density

note that the global spin-flip invariance for $h=0$ implies

$$\langle \sigma_u \rangle_{L,\beta,0} = 0 \quad (2) \quad \text{for any } \beta$$

$$\therefore \quad \lim_{L \uparrow \infty} \lim_{h \downarrow 0} \left\langle \frac{1}{L^d} \sum_{u \in \Lambda_L} \sigma_u \right\rangle_{L,\beta,h} = \lim_{L \uparrow \infty} \lim_{h \downarrow 0} \langle \sigma_u \rangle_{L,\beta,h} = 0 \quad (3)$$

for any β

$m_s(\beta) > 0$ for $\beta > \beta_c$

$\Rightarrow$ the global spin flip symmetry is spontaneously broken!

long-range order (LRO) the model with $h=0$

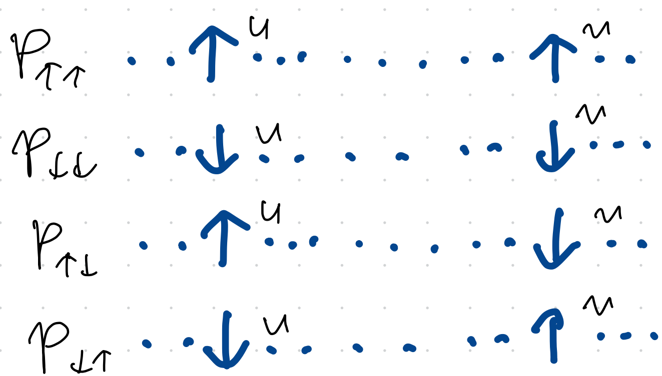
decay property of the two-point correlation function

$\beta<\beta_c$ $\langle\sigma_u\sigma_v\rangle_{L,\beta,0} \sim e^{-\frac{|u-v|}{\xi(\beta)}}$ (1) for $|u-v| \lesssim \frac{L}{2}$

exponential decay with correlation length $0<\xi(\beta)<\infty$

disordered state

$\beta>\beta_c$ $\langle\sigma_u\sigma_v\rangle_{L,\beta,0} \sim (m_s(\beta))^2 > 0$ (2) for large $|u-v| \lesssim \frac{L}{2}$



two spins at distant sites u and v tend to point in the same direction

longr-range order

$P_{\uparrow\uparrow}=P_{\downarrow\downarrow}>P_{\uparrow\downarrow}=P_{\downarrow\uparrow}$

we shall redefine (only here) $\Lambda_L = \{-\frac{L-1}{2}, \cdots, \frac{L-1}{2}\}^d$

Theorem $\langle \sigma_u \sigma_v \rangle_{\beta,h} = \lim_{L\uparrow\infty} \langle \sigma_u \sigma_v \rangle_{L,\beta,h}$ (1) (limit exists)

$\beta < \beta_c$, $\exists\, \xi(\beta) < \infty$ s.t. $\langle \sigma_u \sigma_v \rangle_{\beta,0} \leq e^{-\frac{|u-v|}{\xi(\beta)}}$ (2) for $\forall u, v \in \mathbb{Z}^d$

$\beta > \beta_c$ $\lim_{|u-v|\nearrow\infty} \langle \sigma_u \sigma_v \rangle_{\beta,0} = (m_s(\beta))^2 > 0$ (3)

Summary

ferromagnetic order in the Ising model (with $d \gtrsim 2$, at $\beta > \beta_c$)

is characterized either by

SSB $\quad m_s(\beta) = \lim_{h\downarrow 0} \lim_{L\uparrow\infty} \langle \sigma_u \rangle_{L,\beta,h} > 0$ (4)

LRO $\quad \langle \sigma_u \sigma_v \rangle_{L,\beta,0} \sim \text{const} > 0$ for large $|u-v| \lesssim \frac{L}{2}$ (5)

§ properties of the XY model

(i) Hohenberg–Mermin–Wagner theorem (Hohenberg (1967), Mermin, Wagner (1966))

theorem 1 for $d=1,2$, we have for any $0<\beta<\infty$ that

$$\lim_{h\downarrow 0}\lim_{L\uparrow\infty}\left\langle \frac{1}{L^d}\sum_{u\in\Lambda_L} S_u^{(x)}\right\rangle_{L,\beta,h} = 0 \qquad (1)$$

→ proof part 4

spontaneous magnetization is zero. no SSB

→ essential consequence of the continuous symmetry ($d=2$)

(ii) a high-temperature result (easy)

theorem 2 for any $d=1,2,\cdots$, we have for any $0<\beta<\frac{2}{2d-1}$ that

$$0\le \langle \vec{S}_u\cdot\vec{S}_v\rangle_{L,\beta,0} \le \left(1-\frac{2d-1}{2}\beta\right)^{-1}\left(\frac{2d-1}{2}\beta\right)^{|u-v|} \qquad (2)$$

for any $u,v\in\Lambda_L$ s.t. $|u-v|\le\frac{L}{2}$

→ proof part 5

exponential decay of correlations in the high-temperature region

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(iii) harmonic approximation (not rigorous) Wegner (1967) → part 3

if $\beta \gg 1$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \sim \begin{cases} e^{-\frac{|u-v|}{\xi(\beta)}} & (d=1) \\ |u-v|^{-\frac{1}{2\pi\beta}} & (d=2) \\ \text{const.} > 0 & (d \geq 3) \end{cases} \quad \text{for } 1 \ll |u-v| \lesssim \frac{L}{2} \qquad (1)$$

$d=1$ → no order $\qquad d=3$ → LRO

$d=2$ → power law decay = "exotic" behavior ?

(iii) Kosterlitz-Thouless (KT) transition (not completely rigorous)

there is a phase transition at β_{KT} in the XY model in $d=2$

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{\beta,0} \sim \begin{cases} e^{-\frac{|u-v|}{\xi(\beta)}} & \beta < \beta_{KT} \\ |u-v|^{-\tilde{\eta}(\beta)} & \beta \geq \beta_{KT} \end{cases} \qquad (2)$$

a phase transition without SSB or LRO

early essential works on the KT or BKT or WBKT transition

Wegner (1967) harmonic (or spin-wave) approximation

Berezinskii (1971) harmonic approximation and more

Kosterlitz, Thouless (1972)
complete picture of the transition

Franz Wegner (1940-)

Vadim Berezinskii (1935-1980)

Nobel Prize in Physics 2016

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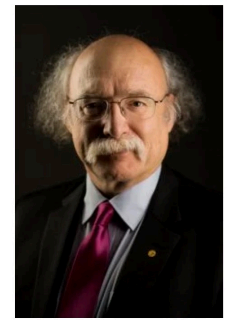
David J. Thouless

Prize share: 1/2 (1934–2019)



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F. Duncan M. Haldane

Prize share: 1/4 (1951–)

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J. Michael Kosterlitz

Prize share: 1/4 (1943–)

The Nobel Prize in Physics 2016 was awarded with one half to David J. Thouless, and the other half to F. Duncan M. Haldane and J. Michael Kosterlitz "for theoretical discoveries of topological phase transitions and topological phases of matter"

The pictures of Wegner by Andreas Mielke.
The picture of Berezinskii from Uspekhi Fizicheskikh Nauk 1981.
The pictures of Thouless, Haldane, and Kosterlitz from the official page of the Nobel Prize.

III rigorous results on the XY model in d=2

our main topic!

McBryan-Spencer theorem — McBryan, Spencer (1977) — proof part 4

theorem 3 for $d=2$, $h=0$, we have for any $0<\beta<\infty$ that

$$0 \leq \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \leq |u-v|^{-\eta(\beta)} \quad (1)$$

for any $u, v \in \Lambda_L$ s.t. $|u-v| \leq \frac{L}{2}$ with $\eta(\beta) > 0$

we also have $\eta(\beta) \simeq (2\beta C)^{-1}$ if $\beta \gg 1$ (C is a constant)

no LRO

Fröhlich-Spencer (1981)

theorem (Fröhlich-Spencer) for $d=2$, $h=0$, sufficiently large β

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \geq |u-v|^{-\tilde{\eta}(\beta)} \quad (2) \quad \text{for } |u-v| \leq \frac{L}{2}$$

rigorously establishes the existence of the (WB)KT transition!!

the proof is difficult

② rigorous results in other dimensions

theorem for $d=1$, $h=0$, we have for any $0<\beta<\infty$ that

$$0 \leq \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \leq e^{-\frac{|u-v|}{\xi(\beta)}} \quad (1) \quad \text{for } \forall u, v \text{ s.t. } |u-v| \leq \frac{L}{2}$$

with $0<\xi(\beta)<\infty$

can be proved, e.g., in the same manner as theorem 3

theorem (Fröhlich-Simon-Spencer, Griffiths)

for $d \geq 3$, we have for sufficiently large β that

LRO $\langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \geq \text{const.} > 0 \quad (2) \quad \text{for any } u, v \in \Lambda_L$

SSB $$\lim_{h \downarrow 0} \lim_{L \uparrow \infty} \left\langle \frac{1}{L^d} \sum_{u \in \Lambda_L} S_u^{(x)} \right\rangle_{L,\beta,h} > 0 \quad (3)$$

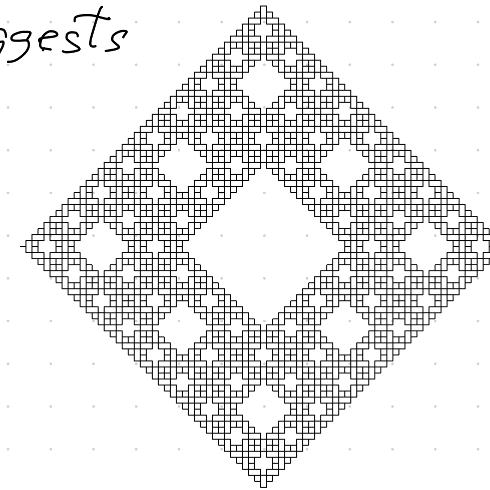
(2) Fröhlich-Simon-Spencer (1976) the reflection positivity method

(2) ⇒ (3) Griffiths (1966)

XY model on a "fractal" lattice with fractal dimension d_F

if $1 < d_F < 2$, the harmonic approximation suggests

$$\langle \vec{S}_u \cdot \vec{S}_v \rangle_{\beta,0} \sim \exp\left[-a(\beta)\,|u-v|^{2-d_F}\right] \quad (1)$$



stretched exponential decay

(slower than exponential decay)

is there another exotic phase transition?

theorem (Koma-Tasaki) if $d_F < 2$, we have for any $0<\beta<\infty$ that

$$0 \le \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \le e^{-\frac{\mathrm{dist}(u,v)}{\xi(\beta)}} \quad (2)$$

with $0 < \xi(\beta) < \infty$

for any u, v s.t. $\mathrm{dist}(u,v) \le \frac{L}{2}$

no phase transitions ...

§ about the Heisenberg model

$$\vec{S}_u = (S_u^{(x)}, S_u^{(y)}, S_u^{(z)}) \in \mathbb{R}^3 \quad \text{s.t. } |\vec{S}_u| = 1$$

$$H_{L,h}(\mathbf{S}) = -\sum_{\{u,v\}\in\mathcal{B}_L} \vec{S}_u \cdot \vec{S}_v - h\sum_{u\in\Lambda_L} S_u^{(x)} \qquad (1)$$

$d=1$ no phase transitions.

Fröhlich–Simon–Spencer + Griffiths

$d \geq 3$ there is a phase transition $\begin{cases} \text{no order for small } \beta \\ \text{LRO and SSB for large } \beta \end{cases}$

the case $d=2$ is still open!!

conjecture (Polyakov) for $d=2$, $h=0$, we have for any $0<\beta<\infty$ that

$$0 \leq \langle \vec{S}_u \cdot \vec{S}_v \rangle_{L,\beta,0} \leq e^{-\frac{|u-v|}{\xi(\beta)}} \quad (1) \quad \text{for } \forall u,v \text{ s.t. } |u-v| \leq \frac{L}{2}$$

with $0 < \xi(\beta) < \infty$

§ plan

part 1 background and main results

part 2 the XY model: the definition and the representation in terms of phases ← short

part 3 Wegner's harmonic approximation

part 4 Mcbryan-Spencer's proof of the absence of order ← main topics

part 5 exponential decay of correlations at high temperatures (appendix)