Kesten, Peierls, Dobrushin

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Griffiths

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Proof of the existence of a phase transition in the two-dimensional Ising model

part 5 low-temperature region

Advanced Topics in Statistical Physics

by Hal Tasaki

theorem 5 there is $\beta_L \in (0,\infty)$ s.t. $m_s(\beta) > 0$ for any $\beta \in (\beta_L, \infty)$ we also have $m_s(\beta) \to 1$ as $\beta \uparrow \infty$

1

§ basic lemma and the proof of theorem 4

lemma let $\beta_L = 0.7$. for $\beta \geq \beta_L$, there is a function $\mu(\beta) > 0$ s.t.

$$\langle \sigma_u \rangle^+_{L,\beta,0} \geq \mu(\beta) \quad (1)$$

for any $u \in \Lambda_L$ and sufficiently large L. we also have $\mu(\beta) \uparrow 1$ as $\beta \uparrow \infty$

part 2 p.5-(3)

$$\frac{\partial}{\partial h} f^+_L(\beta, 0) = -\frac{1}{L^2} \sum_{u \in \Lambda_L} \langle \sigma_u \rangle^+_{L,\beta,0} \leq -\mu(\beta) \quad (2)$$

since $\frac{\partial^2}{\partial h^2} f^+_L(\beta, h) \leq 0$ (3) part 3 p.2

$$\frac{\partial}{\partial h} f^+_L(\beta, h) \leq -\mu(\beta) \quad (4) \quad \text{for } h \geq 0$$

2

for any $h > 0$ and $\beta > \beta_L$ $\quad \leq -\mu(\beta)$

$$f_L^+(\beta,h) - f_L^+(\beta,0) = \int_0^h \boxed{\frac{\partial}{\partial h'} f_L^+(\beta,h')}\, dh' \leq -\mu(\beta)\, h \quad (1)$$

$$\frac{f_L^+(\beta,h) - f_L^+(\beta,0)}{h} \leq -\mu(\beta) \quad (2)$$

$L \uparrow \infty$

$$\frac{f(\beta,h) - f(\beta,0)}{h} \leq -\mu(\beta) \quad (3)$$

$$m_s(\beta) = -\lim_{h \downarrow 0} \frac{f(\beta,h) - f(\beta,0)}{h} \geq \mu(\beta) > 0 \quad (4)$$

§ proof of lemma

(ii) basic idea: Peierls argument

a connected region of $-$ spins in the "sea" of $+$ spins → contains a fixed site

$h=0$

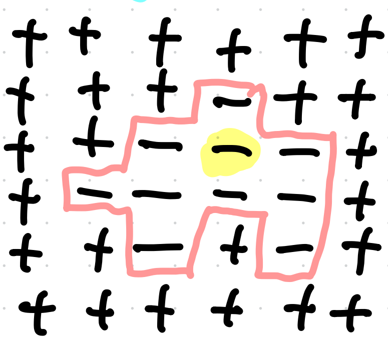
1 dim

$$+\ +\ +\ \overbrace{-\ -\ -\ -\ -\ -}^{n \text{ spins}}\ +\ +\ +\ +$$

energy cost $= 2\times 2 = 4$ penalty $e^{-4\beta}$

n can grow indefinitely → no ferromagnetic order

2 dim



a region surrounded by ℓ bonds

energy cost 2ℓ penalty $e^{-2\beta\ell}$

the number of configurations $\sim 3^{\ell}$

$$e^{-2\beta\ell}\,3^{\ell} = (3e^{-2\beta})^{\ell}$$

large ℓ is surpressed if $3e^{-2\beta} < 1$

→ ferromagnetic order for large β

lattice and the dual lattice

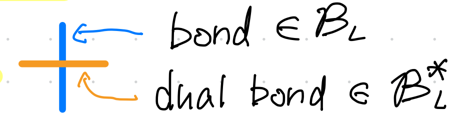
$$\Lambda_L = \{1, \dots, L\}^2, \quad \bar{\Lambda}_L = \{0, 1, \dots, L+1\}^2, \quad \partial\Lambda_L = \bar{\Lambda}_L \setminus \Lambda_L \qquad (1)$$

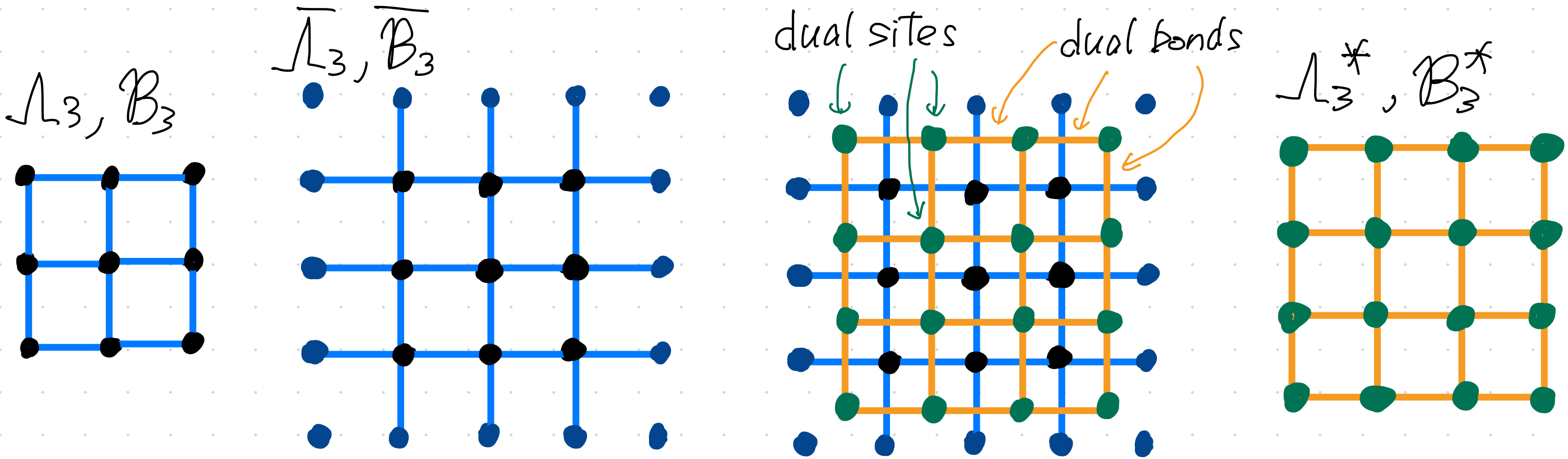
$$\mathcal{B}_L = \{\{u,v\} \mid u,v \in \Lambda_L,\ |u-v| = 1\} \qquad (2)$$

$$\bar{\mathcal{B}}_L = \{\{u,v\} \mid u,v \in \bar{\Lambda}_L, \text{ but not } u,v \in \partial\Lambda_L,\ |u-v| = 1\} \qquad (3)$$

dual site : point at the center of each plaquette (unit square) in $\bar{\Lambda}_L$

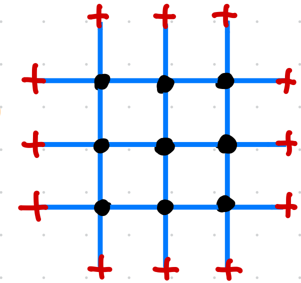
dual bond : bond that crosses a bond in $\bar{\mathcal{B}}_L$





Λ^*_L set of dual sites $(\Lambda^*_L \cong \Lambda_{L+1})$ $\quad$ $\mathcal{B}^*_L$ set of dual bonds $(\mathcal{B}^*_L \cong \mathcal{B}_{L+1})$

representation of $Z_L^+(\beta,0)$



$$H^+_{L,0}(\sigma) = -\sum_{\{u,v\}\in\overline{\mathcal{B}}_L} \sigma_u\sigma_v$$

$\sigma_v = 1$ for $v \in \partial\Lambda_L$

$$Z_L^+(\beta,0) = \sum_{\sigma\in\mathcal{S}_L} e^{-\beta H^+_{L,0}(\sigma)} = \sum_{\sigma\in\mathcal{S}_L}\prod_{\{u,v\}\in\overline{\mathcal{B}}_L} e^{\beta\sigma_u\sigma_v}$$

$$= e^{\beta|\overline{\mathcal{B}}_L|}\sum_{\sigma\in\mathcal{S}_L}\prod_{\{u,v\}\in\overline{\mathcal{B}}_L} \boxed{e^{\beta(\sigma_u\sigma_v-1)}}$$

$$e^{\beta(\sigma_u\sigma_v-1)} = \begin{cases}1, & \sigma_u=\sigma_v\\ e^{-2\beta}, & \sigma_u\neq\sigma_v\end{cases}$$

bond $\{u,v\}$ is happy

bon $\{u,v\}$ is unhappy

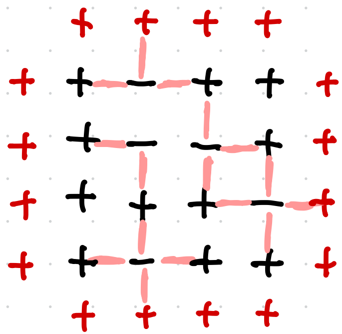
$$= e^{\beta|\overline{\mathcal{B}}_L|}\sum_{\sigma\in\mathcal{S}_L} e^{-2\beta|\mathcal{U}(\sigma)|} \qquad (1)$$

the set of unhappy bonds

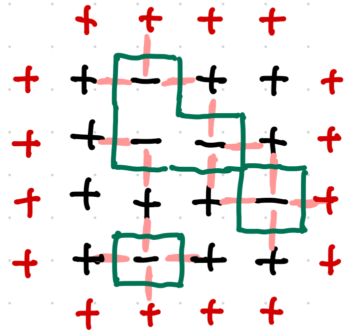
$$\mathcal{U}(\sigma) = \{\{u,v\}\in\overline{\mathcal{B}}_L \mid \sigma_u\neq\sigma_v\}\subset\overline{\mathcal{B}}_L \qquad (2)$$

set of dual bonds corresponding to $\mathcal{U}(\sigma)$

$\mathcal{U}(\sigma)$

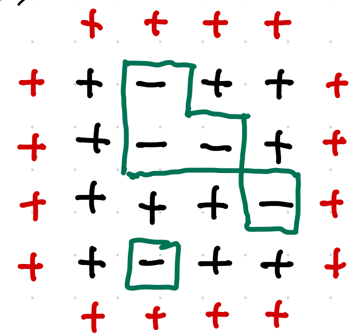


set of unhappy bonds



for each unhappy bond draw its dual bond

$C(\sigma) \subset \mathcal{B}_L^*$



set of dual bonds

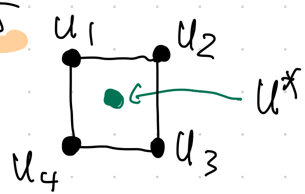
we have

$$\partial C(\sigma) = \emptyset$$

$C \subset \mathcal{B}_L^*$ $\quad n_C(u^*)$: the number of dual bonds in C that contains $u^* \in \Lambda_L^*$

$\partial C = \{u^* \mid n_C(u^*) \text{ is odd}\}$

proof



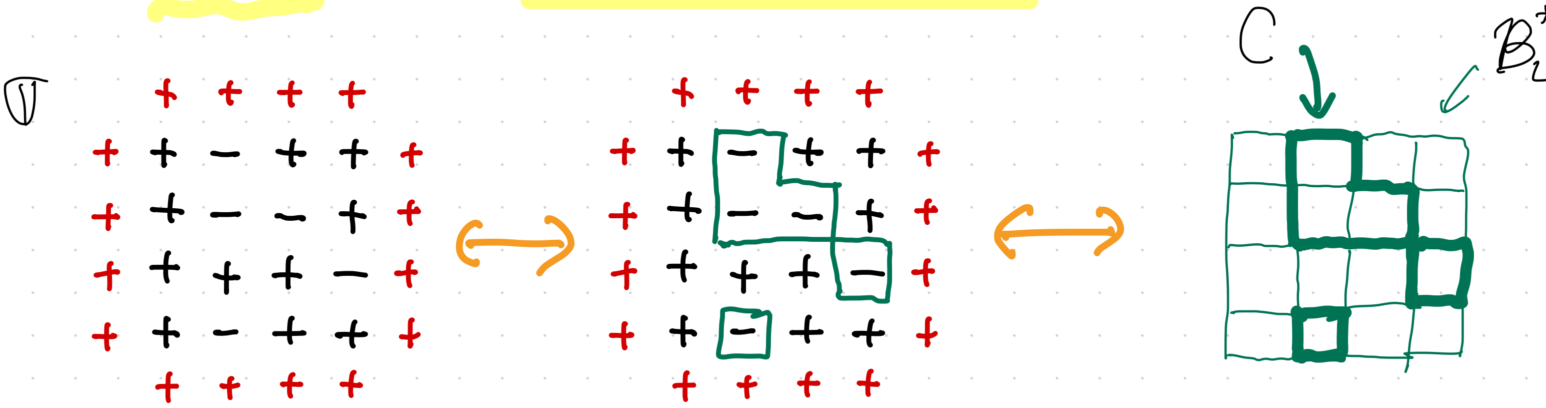
$$(\sigma_{u_1}\sigma_{u_2})(\sigma_{u_2}\sigma_{u_3})(\sigma_{u_3}\sigma_{u_4})(\sigma_{u_4}\sigma_{u_1}) = 1$$

∴ the number of unhappy bonds is even

$n_{C(\sigma)}(u^*)$ is even for any $u^* \in \Lambda_L^*$

one-to-one correspondence between

$\sigma \in \mathcal{S}_L$ and $C \subset \mathcal{B}_L^*$ with $\partial C = \phi$



from p.5-(1) and $|\mathcal{U}(\sigma)| = |C(\sigma)|$

$$Z_L^+(\beta,0) = e^{\beta|\bar{\mathcal{B}}_L|} \sum_{\substack{C \subset \mathcal{B}_L^* \\ (\partial C = \phi)}} e^{-2\beta|C|} \qquad (1)$$

another stochastic geometric representation.

▷ remark: self-duality of the two dimensional Ising model

exactly as in part 4, P5-(4)

$$\mathcal{B}_L^* \simeq \mathcal{B}_{L+1}$$

$$Z_{L+1}^{\text{free}}(\beta,0) = 2^{|\Lambda_{L+1}|}(\cosh\beta)^{|\mathcal{B}_{L+1}|} \boxed{\sum_{\substack{B\subset\mathcal{B}_{L+1}\\(\partial B=\emptyset)}}(\tanh\beta)^{|B|}} \quad (1)$$

define β^* by $\boxed{e^{-2\beta} = \tanh\beta^*}$ (2)

$$e^{-\beta|\overline{\mathcal{B}}_L|} Z_L^+(\beta,0) = 2^{-|\Lambda_{L+1}|}(\cosh\beta^*)^{-|\mathcal{B}_{L+1}|} Z_{L+1}^{\text{free}}(\beta^*,0) \quad (3)$$

$$-\lim_{L\uparrow\infty}\frac{1}{L^2}\log(\cdots)\ \downarrow$$

$$\boxed{\beta f(\beta,0) = \beta^* f(\beta^*,0) + \log(\cosh\beta^*\sinh\beta^*) + \log 2} \quad (4)$$

exact relation that relates the free energy at high and low temperatures!

$\beta\downarrow 0 \longleftrightarrow \beta^*\uparrow\infty$

$\beta\uparrow\infty \longleftrightarrow \beta^*\downarrow 0$

self-dual point $e^{-2\beta_c} = \tanh\beta_c$ (5)

$$\beta_c = \frac{1}{2}\log(\sqrt{2}+1) \simeq 0.44 \quad (6)$$

(transition point)

▷ representation of $\langle \sigma_u \rangle^{+}_{L,\beta,0}$

exacty as P7-(1)

$$Z^{+}_{L}(\beta,0)\langle \sigma_u \rangle^{+}_{L,\beta,0} = \sum_{\sigma \in \mathcal{S}_L} \sigma_u \, e^{-\beta H^{+}_{L,0}(\sigma)} = e^{\beta|\mathcal{B}_L|} \sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} \sigma_u(C)\, e^{-2\beta|C|} \qquad (1)$$

$\sigma_u(C)$ is determined from the one-to-one correspondence between σ and C

(1) and P7-(1)

$$\langle \sigma_u \rangle^{+}_{L,\beta,0} = \frac{\displaystyle\sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} \sigma_u(C)\, e^{-2\beta|C|}}{\displaystyle\sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} e^{-2\beta|C|}} = \sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} \sigma_u(C)\, P(C) \qquad (2)$$

probability that the set C appears

$$P(C) = \frac{e^{-2\beta|C|}}{\displaystyle\sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} e^{-2\beta|C|}} \qquad (3)$$

$$\left(\sum_{\substack{C \subset \mathcal{B}^{*}_{L} \\ (\partial C = \phi)}} P(C) = 1 \qquad (4) \right)$$

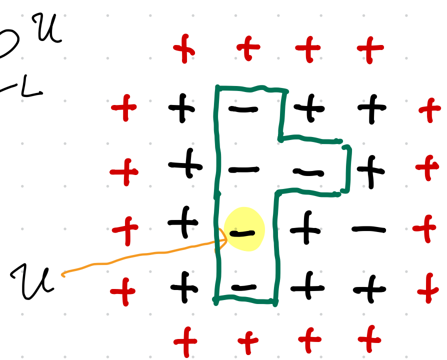
▷ lower bound on $\langle \sigma_u \rangle^{+}_{L,\beta,0}$

$$\mathcal{C}_L = \{ C \mid C \subset \mathcal{B}^*_L,\ \partial C = \phi \} \cong \mathcal{S}_L \quad (1)$$

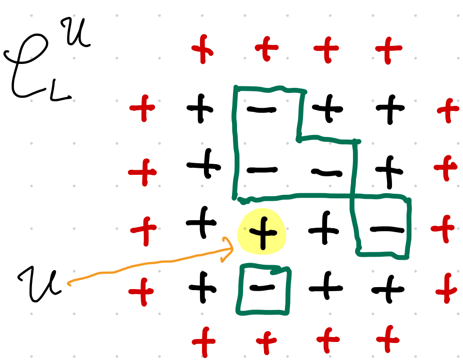
→ one-to-one correspondence in p.7

$$\mathcal{C}^u_L = \{ C \in \mathcal{C}_L \mid u \text{ is surrounded by at least one loop in } C \} \quad (2)$$

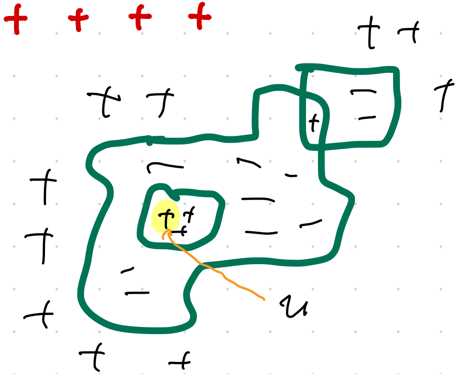
$C \in \mathcal{C}^u_L$



$C \notin \mathcal{C}^u_L$



$$\begin{cases} C \notin \mathcal{C}^u_L \rightarrow \sigma_u(C) = 1 \\ C \in \mathcal{C}^u_L \rightarrow \sigma_u(C) = 1 \text{ or } -1 \end{cases} \quad (3)$$



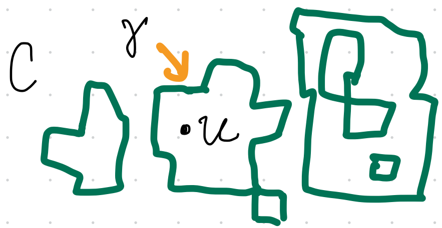
$$\langle \sigma_u \rangle^{+}_{L,\beta,0} = \sum_{C\in\mathcal{C}_L} \sigma_u(C)\, P(C) = \sum_{C\in\mathcal{C}_L^u} \underbrace{\boxed{\sigma_u(C)}}_{\geq -1} P(C) + \sum_{C\in\mathcal{C}_L\setminus\mathcal{C}_L^u} \underbrace{\boxed{\sigma_u(C)}}_{=\,1} P(C)$$

$$\geq -\sum_{C\in\mathcal{C}_L^u} P(C) + \underbrace{\boxed{\sum_{C\in\mathcal{C}_L\setminus\mathcal{C}_L^u} P(C)}}_{=\, 1-\sum_{C\in\mathcal{C}_L^u} P(C)} = 1 - 2\sum_{C\in\mathcal{C}_L^u} P(C) \qquad (1)$$

We shall upper-bound $\sum_{C\in\mathcal{C}_L} P(C)$

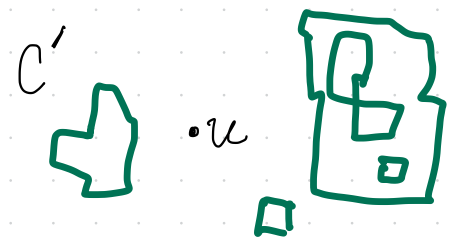
a loop with no self-intersections

if $C\in\mathcal{C}_L^u$ there is a contour $\gamma\subset C$ that surrounds u



$C' = C\setminus\gamma$

$\partial C' = \phi$



from p9-(3)

$$P(C) = P(C'\cup\gamma) = e^{-2\beta|\gamma|} P(C') \qquad (2)$$

$$\sum_{C \in \mathcal{C}_L^u}' P(C) \leq \sum_{\substack{\gamma:\text{ contour} \\ \text{surrounding } u}}' \sum_{\substack{C' \in \mathcal{C}_L \\ (C' \cap \gamma = \phi)}}' e^{-2\beta|\gamma|} P(C')$$

$$\leq \sum_{\gamma}' \sum_{C' \in \mathcal{C}_L}' e^{-2\beta|\gamma|} P(C') = \sum_{\gamma}' e^{-2\beta|\gamma|} = \sum_{\ell=4}^{\infty}{}' W_\ell \, e^{-2\beta\ell} \quad (1)$$

W_ℓ : the number of contours with ℓ bonds that surround u

we shall show $W_\ell \leq \frac{1}{72} \ell \, 3^\ell$ (2)

$$\langle \sigma_u \rangle^+_{L,\beta,0} \geq 1 - 2 \sum_{C \in \mathcal{C}_L^u}' P(C) \geq 1 - 2 \sum_{\ell=4}^{\infty}{}' W_\ell \, e^{-2\beta\ell}$$

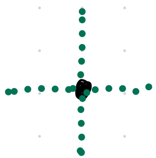
$$\geq 1 - 2 \boxed{\sum_{\ell=4}^{\infty}{}' \frac{\ell (3 e^{-2\beta})^\ell}{72}} =: \mu(\beta) > 0 \quad (3)$$

$< \frac{1}{2}$ for $\beta \geq \beta_L = 0.7$

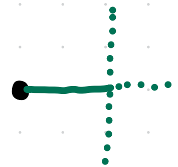
clearly $\mu(\beta) \uparrow 1$ as $\beta \uparrow \infty$

iii upper bound on W_ℓ

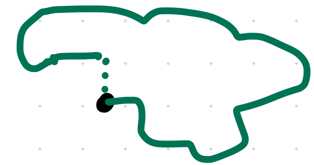
W_ℓ' : the number of contours with ℓ bonds that contain a fixed site



first step 4 choices



2nd ~ $(\ell-1)$-th step
at most 3 choices



ℓ-th step at most 1 choice

$$W_\ell' \leq 4 \times 3^{\ell-2} \times \boxed{\frac{1}{2}}$$

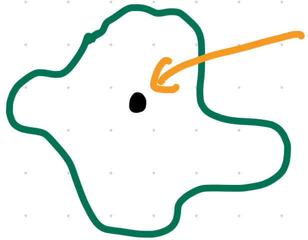
double counting

W_ℓ'' : the number of possible patterns of contours with ℓ bonds
(identify two contours that are related by a translation)

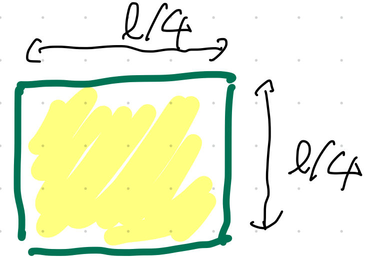
the starting point can be anywhere

$$W_\ell'' = \boxed{\frac{1}{\ell}} W_\ell' \leq \frac{2\, 3^{\ell-2}}{\ell}$$

W_ℓ : the number of contours with ℓ bonds that surround u



u can be any where



the number of sites surrounded by a contour with ℓ bonds $\leq \left(\frac{\ell}{4}\right)^2$ (1)

$$W_\ell \leq \left(\frac{\ell}{4}\right)^2 W_\ell'' \leq \left(\frac{\ell}{4}\right)^2 \frac{2\,3^{\ell-2}}{\ell} = \frac{\ell\,3^\ell}{72} \quad (2)$$