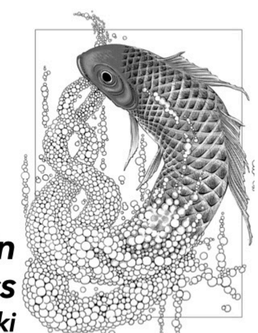
Proof of the existence of a phase transition in the two-dimensional Ising model

part 4 high-temperature region

Advanced Topics in Statistical Physics
by Hal Tasaki



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theorem 4 there is $\beta_H \in (0,\infty)$ s.t. $\tanh \beta_H = \frac{1}{3}$

$m_s(\beta) = 0$ for any $\beta \in (0, \beta_H)$

§ basic lemma and the proof of theorem 4

we may use free b.c.

lemma if $\beta > 0$ satisfies $3 \tanh \beta < 1$ (1) then

$$\langle \sigma_u \rangle^{per}_{L,\beta,h} \leq \frac{4}{3} \frac{\tanh \beta h}{1 - 3\tanh\beta} \leq \frac{4\beta h}{3(1-3\tanh\beta)} \quad (2)$$

for any $h \geq 0$, $u \in \Lambda_L$ and L

note first that

concavity (part3, p.2) symmetry (part3, p3-(4))

$$f_L^{per}(\beta, 0) \geq \frac{1}{2}\{ f_L^{per}(\beta, h) + f_L^{per}(\beta, -h) \} = f_L^{per}(\beta, h) \quad (3)$$

$$\therefore \quad f_L^{per}(\beta, h) - f_L^{per}(\beta, 0) \leq 0 \quad (4)$$

for $h > 0$, $3\tanh\beta < 1$ part2, p.5-(3) 2

$$\frac{\partial}{\partial h} f_L^{per}(\beta, h) = -\frac{1}{L^2}\sum_{u\in\Lambda_L}\langle\sigma_u\rangle_{L,\beta,h}^{per} \geq -\frac{4\beta h}{3(1-3\tanh\beta)} \quad (1)$$

$$f_L^{per}(\beta,h) - f_L^{per}(\beta,0) = \int_0^h \frac{\partial}{\partial h'} f_L^{per}(\beta,h')\,dh'$$

$$\geq -\int_0^h \frac{4\beta h'}{3(1-3\tanh\beta)}\,dh' = -\frac{2\beta}{3(1-3\tanh\beta)}h^2 \quad (2)$$

$$0 \overset{\text{p.1-(4)}}{\geq} \frac{1}{h}\{f_L^{per}(\beta,h) - f_L^{per}(\beta,0)\} \overset{(2)}{\geq} -\frac{2\beta}{3(1-3\tanh\beta)}h \quad (3)$$

$L \nearrow \infty$

$$0 \geq \frac{1}{h}\{f(\beta,h) - f(\beta,0)\} \geq -\frac{2\beta}{3(1-3\tanh\beta)}h \quad (4)$$

$h \downarrow 0$

$$0 \geq \frac{\partial}{\partial h_+} f(\beta,0) \geq 0 \quad (5) \qquad m_s(\beta) = -\frac{\partial}{\partial h_+} f(\beta,0) = 0 \quad (6)$$

§ proof of lemma

Fisher, 1967

Michael Fisher (1931– 2021)

stochastic geometric representation of $\langle \sigma_u \rangle^{per}_{\beta,L,h}$

We only use the periodic boundary condition for a fixed L

$$\Lambda_L \to \Lambda, \quad \mathcal{B}^{per}_L \to \mathcal{B}, \quad \mathcal{S}_L \to \mathcal{S}$$

$$H^{per}_{L,h}(\sigma) \to H_h(\sigma) = -\sum_{\{u,v\}\in\mathcal{B}} \sigma_u \sigma_v - h \sum_{u\in\Lambda} \sigma_u, \quad Z^{per}_L(\beta,h) \to Z(\beta,h)$$

basic observation

$$\boxed{\begin{aligned} a &= \tanh\beta \\ b &= \tanh\beta h \end{aligned}}$$

$$e^{-\beta H_h(\sigma)} = \prod_{\{u,v\}\in\mathcal{B}} e^{\beta\sigma_u\sigma_v} \prod_{u\in\Lambda} e^{\beta h \sigma_u}$$

$$= (\cosh\beta)^{|\mathcal{B}|} (\cosh\beta h)^{|\Lambda|} \prod_{\{u,v\}\in\mathcal{B}} (1 + a\,\sigma_u\sigma_v) \prod_{u\in\Lambda} (1 + b\,\sigma_u) \qquad (1)$$

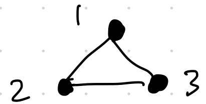
because in general $\tau = \pm 1$, $\alpha \in \mathbb{R}$ imply

$$e^{\alpha\tau} = \cosh\alpha + \tau\sinh\alpha = \cosh\alpha\,(1 + \tau\tanh\alpha) \qquad (2)$$

expansion of the interaction term

$$\prod_{\{u,v\}\in\mathcal{B}} (1 + a\,\sigma_u\sigma_v) = \sum_{B\subset\mathcal{B}} a^{|B|} \prod_{\{u,v\}\in B} \sigma_u\sigma_v$$

all subsets of $\mathcal{B}$

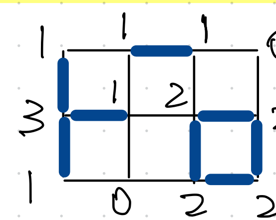


$$(1+a\sigma_1\sigma_2)(1+a\sigma_2\sigma_3)(1+a\sigma_3\sigma_1)$$

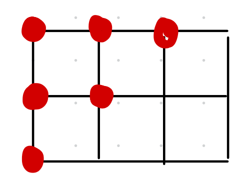
$$= 1 + a\sigma_1\sigma_2 + a\sigma_2\sigma_3 + a\sigma_3\sigma_1 + a^2 \overset{\sigma_3\sigma_1}{\boxed{\sigma_1\sigma_2\sigma_2\sigma_3}} + a^2 \overset{\sigma_1\sigma_2}{\boxed{\sigma_2\sigma_3\sigma_3\sigma_1}} + a^2 \overset{\sigma_2\sigma_3}{\boxed{\sigma_3\sigma_1\sigma_1\sigma_2}} + a^3 \overset{1}{\boxed{\sigma_1\sigma_2\sigma_2\sigma_3\sigma_3\sigma_1}}$$

$$= \sum_{B\subset\mathcal{B}} a^{|B|} \prod_{u\in\Lambda} \sigma_u^{\,n_B(u)}$$

$n_B(u)$: the number of bonds in B that contains u



$$= \sum_{B\subset\mathcal{B}} a^{|B|} \prod_{u\in\partial B} \sigma_u \qquad (1)$$



$$\partial B = \{ u\in\Lambda \mid n_B(u) \text{ is odd} \} \qquad (2)$$

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(iii) Stochastic geometric representation of $Z(\beta,0)$

$$Z(\beta,0) = \sum_{\sigma\in\mathcal{S}} e^{-\beta H_0(\sigma)} = \sum_{\sigma\in\mathcal{S}} (\cosh\beta)^{|\mathcal{B}|} \sum_{B\subset\mathcal{B}} a^{|B|} \prod_{u\in\partial B} \sigma_u \qquad (1)$$

since $\sum_{\sigma=\pm1} \sigma = 0, \quad \sum_{\sigma=\pm1} 1 = 2 \qquad (2)$

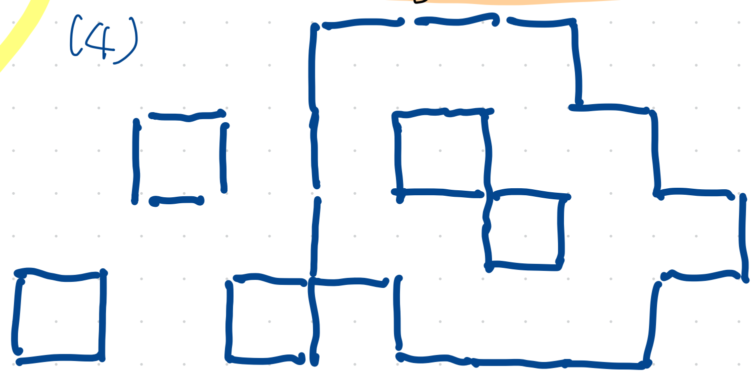
$$\sum_{\sigma\in\mathcal{S}} \prod_{u\in\partial B} \sigma_u = \left(\prod_{u\in\partial B} \sum_{\sigma_u=\pm1} \sigma_u\right)\left(\prod_{u\in\Lambda\setminus\partial B} \sum_{\sigma_u=\pm1} 1\right) = \begin{cases} 0, & \partial B\neq\phi \\ 2^{|\Lambda|}, & \partial B=\phi \end{cases} \qquad (3)$$

$\therefore$

$$Z(\beta,0) = 2^{|\Lambda|}(\cosh\beta)^{|\mathcal{B}|} \boxed{\sum_{\substack{B\subset\mathcal{B} \\ (\partial B=\phi)}} a^{|B|}} \qquad (4)$$

$\Leftarrow$ weighted sum over geometric objects

$B\subset\mathcal{B}$ s.t. $\partial B=\phi$ $\rightsquigarrow$



iii) Stochastic geometric representation of $Z(\beta,h)$

magnetic field term $\prod_{u\in\Lambda}(1+b\sigma_u) = \sum_{S\subset\Lambda} b^{|S|} \prod_{v\in S}\sigma_v$ (1)

all subsets of Λ

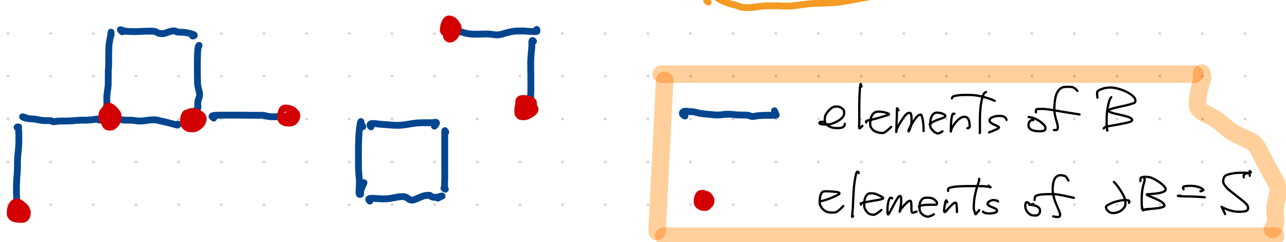
$$Z(\beta,h) = \sum_{\sigma\in\mathcal{X}} (\cosh\beta)^{|\mathcal{B}|} (\cosh\beta h)^{|\Lambda|} \sum_{B\subset\mathcal{B}} \sum_{S\subset\Lambda} a^{|B|} b^{|S|} \Big(\prod_{u\in\partial B}\sigma_u\Big)\Big(\prod_{v\in S}\sigma_v\Big) \quad (2)$$

$$\sum_{\sigma\in\mathcal{X}} \Big(\prod_{u\in\partial B}\sigma_u\Big)\Big(\prod_{v\in S}\sigma_v\Big) = \begin{cases} 0, & \partial B\neq S \\ 2^{|\Lambda|}, & \partial B = S \end{cases} \quad (3)$$

$$Z(\beta,h) = (\cosh\beta)^{|\mathcal{B}|}(2\cosh\beta h)^{|\Lambda|} \sum_{B\subset\mathcal{B}} a^{|B|}\, b^{|\partial B|} \quad (4)$$

sum over all subsets

$a^{13}\, b^{7}$



(iii) Stochastic geometric representation of $\langle \sigma_u \rangle_{\beta,h}$

u : any fixed site

$$Z(\beta,h)\langle \sigma_u \rangle_{\beta,h} = \sum_{\sigma\in\mathcal{S}} \sigma_u e^{-\beta H_h(\sigma)}$$

$$= (\cosh\beta)^{|\mathcal{B}|}(\cosh\beta h)^{|\Lambda|} \sum_{B\subset\mathcal{B}} \sum_{S\subset\Lambda} a^{|B|} b^{|S|} \sum_{\sigma\in\mathcal{S}} \Big(\sigma_u \prod_{v\in\partial B}\sigma_v \prod_{w\in S}\sigma_w\Big)$$

the sum is nonzero and $2^{|\Lambda|}$ when

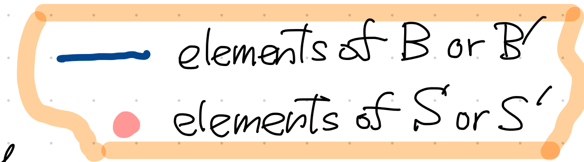
(i) $u\in S$, $u\notin\partial B$, $\partial B = S\setminus\{u\}$

(ii) $u\notin S$, $u\in\partial B$, $\partial B\setminus\{u\} = S$

$$= (\cosh\beta)^{|\mathcal{B}|}(2\cosh\beta h)^{|\Lambda|} \sum_{\substack{B\subset\mathcal{B}\\ S\subset\Lambda\\ ((i)\text{ or }(ii))}} a^{|B|} b^{|S|} \qquad (1)$$

(iii) construction of random walks from B, S satisfying (i) or (ii)

n-step walk $\omega=(\omega_0,\omega_1,\dots,\omega_n)$, $\omega_j\in\Lambda$, $\{\omega_j,\omega_{j+1}\}\in B$

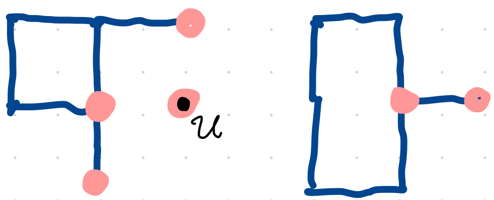


(i)

$u\in S$

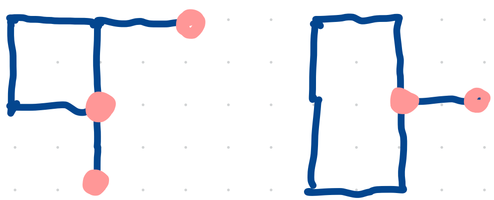
$u\notin\partial B$

$\partial B=S\setminus\{u\}$



$\omega=(u)$

$n=0$, $\omega_0=u$



set $B'=B$

$S'=S\setminus\{u\}$

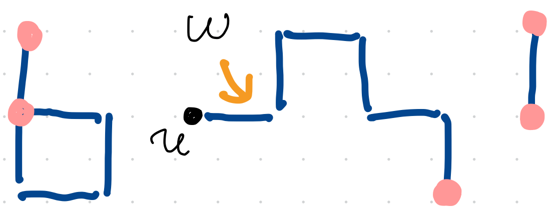
$\partial B'=S'$

(ii)

$u\notin S$

$u\in\partial B$

$\partial B\setminus\{u\}=S$



$\omega_0=u$, $\{\omega_j,\omega_{j+1}\}\in B$, $\omega_n\in S$

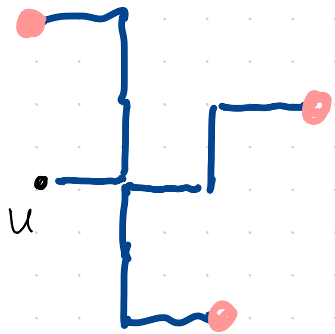
set $B'=B\setminus\omega$

$S'=S\setminus\{\omega_n\}$

$\partial B'=S'$

remark

the choice of walk may not be unique



$\Leftarrow$ there are three possible ω

not a problem since we will prove an upper bound

P.7-(1)

$$Z(\beta,h)\langle\sigma_u\rangle_{\beta,h} = (\cosh\beta)^{|\mathcal{B}|}(2\cosh\beta h)^{|\Lambda|}\sum_{\substack{B\subset\mathcal{B}\\ S\subset\Lambda\\ ((i)\text{ or }(ii))}}{}' a^{|B|}b^{|S|}$$

$$\leq (\cosh\beta)^{|\mathcal{B}|}(2\cosh\beta h)^{|\Lambda|}\sum_{n=0}^{\infty}{}'\sum_{\substack{w:u\rightsquigarrow\\ (n\text{ step walks})}}{}' a^n b \sum_{\substack{B'\subset\mathcal{B}\\ S'\subset\Lambda\\ \left(\begin{subarray}{l}\partial B'=S'\\ B'\cap w=\phi\\ S'\not\ni w_n\end{subarray}\right)}}{}' a^{|B'|}b^{|S'|}$$

$$\leq \sum_{\substack{B'\subset\mathcal{B}\\ S'\subset\Lambda'\\ (\partial B'=S')}}{}' a^{|B'|}b^{|S'|} = \sum_{B'\subset\mathcal{B}} a^{|B'|}b^{|\partial B'|}$$

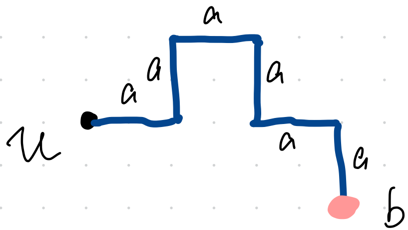
P.6-(4)

$$\leq Z(\beta,h)\sum_{n=0}^{\infty}{}'\sum_{\substack{w:u\rightsquigarrow\\ (n\text{ step walks})}}{}' a^n b$$

(1)

$$\boxed{\langle \sigma_u \rangle_{\beta,h} \le \sum_{n=0}^{\infty} \sum_{\substack{\omega: u \rightsquigarrow \\ (n \text{ step walks})}} a^n b}$$

$$\boxed{\begin{aligned} a &= \tanh\beta \\ b &= \tanh\beta h \end{aligned}}$$



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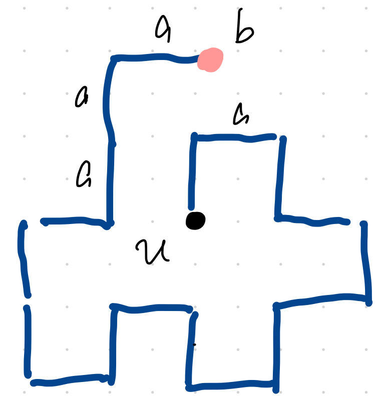


, . . .

$$a^6 b = (\tan\beta)^6 \tanh\beta h$$

$$a^{18} b = (\tanh\beta)^{18} \tanh\beta h$$

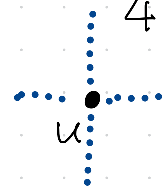
$$\langle \sigma_u \rangle_{\beta,h} \leq \sum_{n=0}^{\infty} \sum_{\substack{\omega: u \rightsquigarrow \\ (n \text{ step walks})}} a^n b = \sum_{n=0}^{\infty} W_n a^n b \quad (1)$$

$a = \tanh\beta$
$b = \tanh\beta h$

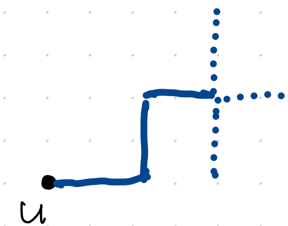
W_n : the number of n-step walks

$$W_n \leq 4 \times 3^{n-1} \quad (2)$$

1st step
4 choices



2nd ~ n-th step
at most 3 choices



thus

$$\langle \sigma_u \rangle_{\beta,h} \leq \sum_{n=0}^{\infty} 4\, 3^{n-1} a^n b = \frac{4}{3} b \sum_{n=0}^{\infty} (3a)^n = \frac{4}{3} b \frac{1}{1-3a}$$

$$= \frac{4}{3} \frac{\tanh\beta h}{1 - 3\tanh\beta} \quad (3)$$