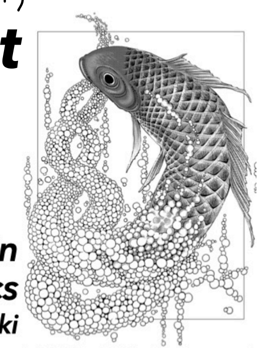
Proof of the existence of a phase transition in the two-dimensional Ising model

or the thermodynamic limit

part 3 the existence of the infinite volume limit



Advanced Topics in Statistical Physics
by Hal Tasaki

theorem 3 for any $\beta > 0$ and $h \in \mathbb{R}$, the limit

$$f(\beta,h) = \lim_{L\uparrow\infty} f_L^{BC}(\beta,h) \quad (1)$$

exsits, and is independent of BC = free, per, +

$f(\beta,h)$ is concave in h and satisfies $f(\beta,h) = f(\beta,-h)$

§ concavity and symmetry

we prove concavity and symmetry, assuming the existence/uniqueness of the limit

△ concavity

for BC = free, per, +

part 2, p.5-(3) $$\frac{\partial}{\partial h} f_L^{BC}(\beta,h) = -\langle m(\sigma)\rangle_{L,\beta,h}^{BC} \quad (1)$$

with $$m(\sigma) = \frac{1}{L^2}\sum_{u\in\Lambda_L} \sigma_u \quad (2)$$

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$$\frac{\partial^2}{\partial h^2} f_L^{BC}(\beta,h) = -\frac{\partial}{\partial h}\langle m(\sigma)\rangle^{BC}_{L,\beta,h} = -\frac{\partial}{\partial h}\frac{\sum_{\sigma\in\mathcal{S}_L} m(\sigma)\, e^{-\beta H^{BC}_{L,h}(\sigma)}}{Z_L^{BC}(\beta,h)}$$

$$= -\frac{\sum_{\sigma} m(\sigma)\overbrace{\boxed{\left\{-\beta\frac{\partial}{\partial h}H^{BC}_{L,h}(\sigma)\right\}}}^{\beta L^2 m(\sigma)} e^{-\beta H^{BC}_{L,h}(\sigma)}}{Z_L^{BC}(\beta,h)} + \frac{\left(\sum_{\sigma} m(\sigma) e^{-\beta H^{BC}_{L,h}(\sigma)}\right)\overbrace{\boxed{\frac{\partial}{\partial h}Z_L^{BC}(\beta,h)}}^{\beta L^2\sum_{\sigma} m(\sigma) e^{-\beta H^{BC}_{L,h}(\sigma)}}}{\left\{Z_L^{BC}(\beta,h)\right\}^2}$$

part 2 - p.5 - (1)

$$= \beta L^2\left\{-\langle (m(\sigma))^2\rangle^{BC}_{L,h} + \left\{\langle m(\sigma)\rangle^{BC}_{L,h}\right\}^2\right\}$$

$$= -\beta L^2\left\langle \left\{m(\sigma) - \langle m(\sigma)\rangle^{BC}_{L,h}\right\}^2\right\rangle^{BC}_{L,h} \leq 0 \qquad (1)$$

definition of concavity part 1 - p.1

$f_L^{BC}(\beta,h)$ is concave in h $\longrightarrow$ so is the $L\uparrow\infty$ limit $f(\beta,h)$

iii) symmetry

for BC = free, per (but not for +)

$$H^{BC}_{L,h}(-\sigma) = H^{BC}_{L,-h}(\sigma) \quad (1) \qquad \text{where } -\sigma = (-\sigma_u)_{u\in\Lambda_L} \quad (2)$$

see part 2, p3, (3), (4)

$$Z^{BC}_L(\beta,h) = \sum_{\sigma\in\mathcal{S}_L} e^{-\beta H^{BC}_{L,h}(\sigma)} = \sum_{\sigma\in\mathcal{S}_L} e^{-\beta H^{BC}_{L,h}(-\sigma)}$$

$$= \sum_{\sigma\in\mathcal{S}_L} e^{-\beta H^{BC}_{L,-h}(\sigma)} = Z^{BC}_L(\beta,-h) \quad (3)$$

Thus $f^{BC}_L(\beta,h) = f^{BC}_L(\beta,-h)$ (4) $\xrightarrow{L\uparrow\infty}$ $f(\beta,h) = f(\beta,-h)$ (5)

(note: $f^{+}_L(\beta,h)$ does not have the symmetry

the symmetry emerges (or is restored) in the limit $L\uparrow\infty$)

§ independence on the boundary conditions

We assume $f(\beta,h) = \lim_{L\uparrow\infty} f_L^{\text{free}}(\beta,h)$ (1) exists

and prove $\lim_{L\uparrow\infty} f_L^{BC}(\beta,h) = f(\beta,h)$ (2) for BC = per, +

part 2, p.3

$$H_{L,h}^{\text{free}}(\sigma) = -\sum_{\{u,v\}\in\mathcal{B}_L}' \sigma_u\sigma_v - h\sum_{u\in\Lambda_L}' \sigma_u \quad (3)$$

$$H_{L,h}^{\text{per}}(\sigma) = -\sum_{\{u,v\}\in\mathcal{B}_L^{\text{per}}}' \sigma_u\sigma_v - h\sum_{u\in\Lambda_L}' \sigma_u \quad (4)$$

$$H_{L,h}^{+}(\sigma) = -\sum_{\{u,v\}\in\overline{\mathcal{B}}_L}' \sigma_u\sigma_v - h\sum_{u\in\Lambda_L}' \sigma_u \quad (5)$$

$$|H_{L,h}^{\text{per}}(\sigma) - H_{L,h}^{\text{free}}(\sigma)| \le |\mathcal{B}_L^{\text{per}}\setminus\mathcal{B}_L| = 2L \quad (6)$$

$$|H_{L,h}^{+}(\sigma) - H_{L,h}^{\text{free}}(\sigma)| \le |\overline{\mathcal{B}}_L\setminus\mathcal{B}_L| = 4L \quad (7) \quad \text{for any } \sigma$$

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for BC = per, +

$$Z_L^{BC}(\beta,h) = \sum_{\sigma} e^{-\beta H_{L,h}^{BC}(\sigma)} \leq \sum_{\sigma} e^{-\beta H_{L,h}^{free}(\sigma) + 4\beta L}$$

$$= e^{4\beta L} Z_L^{free}(\beta,h) \qquad (1)$$

similarly $Z_L^{BC}(\beta,h) \geq e^{-4\beta L} Z_L^{free}(\beta,h)$ (2)

since $f_L^{BC}(\beta,h) = -\frac{1}{\beta L^2} \log Z_L^{BC}(\beta,h)$ (3)

$$f_L^{free}(\beta,h) - \frac{4}{L} \leq f_L^{BC}(\beta,h) \leq f_L^{free}(\beta,h) + \frac{4}{L} \qquad (4)$$

$$\therefore \boxed{\lim_{L\uparrow\infty} f_L^{BC}(\beta,h) = f(\beta,h)} \qquad (5)$$

(remark one can treat many other boundary conditions similarly)

§ the existence of the infinite volume limit for the free boundary conditions

lemma 1 for any $\beta>0$, $h\in\mathbb{R}$ and $L\geq 1$

$$\left|f_{2L}^{\text{free}}(\beta,h)-f_{L}^{\text{free}}(\beta,h)\right|\leq\frac{1}{L}\qquad(1)$$

proof next page

fix β, h and let $g_n = f_{2^n}^{\text{free}}(\beta,h)$ (2)

$$(1)\Rightarrow |g_{n+1}-g_n|\leq\frac{1}{2^n}\qquad(3)$$

for $m>n$, $\displaystyle |g_m-g_n|\leq\sum_{j=n}^{m-1}|g_{j+1}-g_j|\leq\sum_{j=n}^{m-1}\frac{1}{2^j}\leq\frac{1}{2^{n-1}}$ (4)

$(g_n)_{n=1,2,\ldots}$ is a Cauchy sequence $\Rightarrow$ $\lim_{n\uparrow\infty} g_n$ exists!

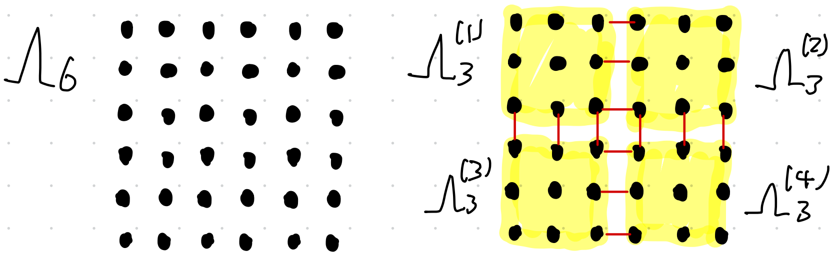
we thus find

define $f(\beta,h)$

lemma 2 for any $\beta>0$ and $h\in\mathbb{R}$, $\lim_{n\uparrow\infty} f_{2^n}^{\text{free}}(\beta,h)$ exists

proof of lemma 1

decompose Λ_{2L} into four copies of Λ_L



$$\Lambda_{2L} = \Lambda_L^{(1)} \cup \Lambda_L^{(2)} \cup \Lambda_L^{(3)} \cup \Lambda_L^{(4)} \tag{1}$$

decomposition of spin configurations

$$\sigma = (\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}, \sigma^{(4)}) \tag{2}$$

decomposition of the Hamiltonian

$$H_{2L}^{\text{free}}(\sigma) = \sum_{j=1}^{4} H_L^{(j),\text{free}}(\sigma^{(j)}) + (\text{interactions on the cuts}) \tag{3}$$

$$\left| H_{2L}^{\text{free}}(\sigma) - \sum_{j=1}^{4} H_L^{(j),\text{free}}(\sigma^{(j)}) \right| \leq 4L \quad \text{for any } \sigma \tag{4}$$

$$Z_{2L}^{\text{free}}(\beta,h) = \sum_{\sigma\in\mathcal{S}_{2L}} e^{-\beta H_{2L,h}^{\text{free}}(\sigma)} \le \sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)},\sigma^{(4)}\in\mathcal{S}_L} e^{-\beta\sum_{j=1}^{4} H_{L,h}^{(j),\text{free}}(\sigma^{(j)}) + 4\beta L}$$

$$= e^{4\beta L}\prod_{j=1}^{4}\left(\sum_{\sigma^{(j)}\in\mathcal{S}_L} e^{-\beta H_{L,h}^{(j),\text{free}}(\sigma^{(j)})}\right) = e^{4\beta L}\left\{Z_L^{\text{free}}(\beta,h)\right\}^4 \qquad (1)$$

similarly

$$Z_{2L}^{\text{free}}(\beta,h) \ge e^{-4\beta L}\left\{Z_L^{\text{free}}(\beta,h)\right\}^4 \qquad (2)$$

we thus have

$$f_{2L}^{\text{free}}(\beta,h) = -\frac{1}{4L^2\beta}\log Z_{2L}^{\text{free}}(\beta,h)$$

$$\left\{\begin{aligned} &\ge -\frac{1}{4L^2\beta}\left\{4\beta L + 4\log Z_L^{\text{free}}(\beta,h)\right\} = f_L^{\text{free}}(\beta,h)\boxed{-\frac{1}{L}} \\ &\le -\frac{1}{4L^2\beta}\left\{-4\beta L + 4\log Z_L^{\text{free}}(\beta,h)\right\} = f_L^{\text{free}}(\beta,h)\boxed{+\frac{1}{L}} \end{aligned}\right. \qquad (3)$$

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final step of the proof

use lemma 2 to show $\lim_{L\to\infty} f_L^{\text{free}}(\beta, h)$ exists

two decompositions $\Lambda_{2^n L}$ $\longrightarrow$ (i) $(2^n)^2$ copies of Λ_L

$\longrightarrow$ (ii) L^2 copies of Λ_{2^n}

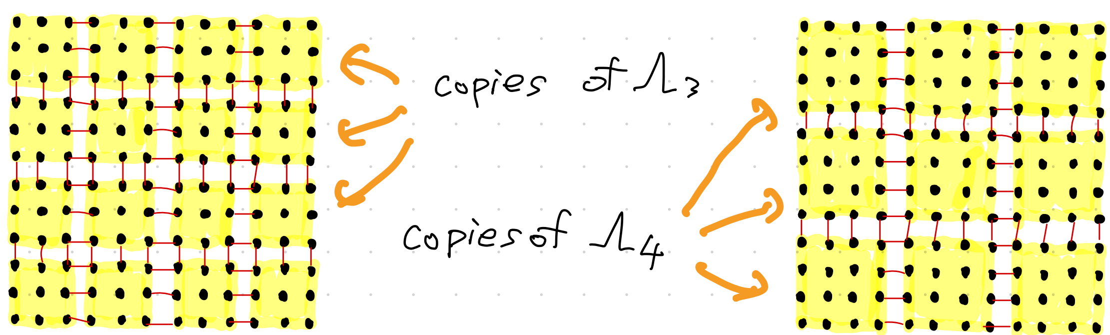
decomposition of the Hamiltonian

$$H^{\text{free}}_{2^n L, h}(\sigma) = \begin{cases} \displaystyle\sum_{j=1}^{(2^n)^2} H^{(j),\text{free}}_{L,h}(\sigma^{(j)}) + \cdots & \text{(i)} \\ \displaystyle\sum_{k=1}^{L^2} H^{(k),\text{free}}_{2^n,h}(\sigma^{(k)}) + \cdots & \text{(ii)} \end{cases}$$

(1)

$L=3,\ n=2$

Λ_{12}



(i) (ii)

$$\left| H^{\text{free}}_{2^nL,h}(\sigma) - \sum_{j=1}^{(2^n)^2} H^{(j),\text{free}}_{L,h}(\sigma^{(j)}) \right| \leq 2(2^n-1)(2^nL) \qquad (1)$$

$$\left| H^{\text{free}}_{2^nL,h}(\sigma) - \sum_{k=1}^{L^2} H^{(k),\text{free}}_{2^n,h}(\sigma^{(k)}) \right| \leq 2(L-1)(2^nL) \qquad (2)$$

$$\left| f^{\text{free}}_{2^nL}(\beta,h) - f^{\text{free}}_{L}(\beta,h) \right| \leq \frac{2(2^n-1)(2^nL)}{(2^nL)^2} \leq \frac{2}{L} \qquad (3)$$

$$\left| f^{\text{free}}_{2^nL}(\beta,h) - f^{\text{free}}_{2^n}(\beta,h) \right| \leq \frac{2(L-1)(2^nL)}{(2^nL)^2} \leq \frac{1}{2^{n-1}} \qquad (4)$$

$$\left| f^{\text{free}}_{L}(\beta,h) - f^{\text{free}}_{2^n}(\beta,h) \right| \leq \frac{2}{L} + \frac{1}{2^{n-1}} \qquad (5)$$

$n \nearrow \infty$ and use lemma 2

$$\left| f^{\text{free}}_{L}(\beta,h) - f(\beta,h) \right| \leq \frac{2}{L} \quad (6) \Rightarrow \boxed{\lim_{L \nearrow \infty} f^{\text{free}}_{L}(\beta,h) = f(\beta,h)} \quad (7)$$