Proof of the existence of a phase transition in the two-dimensional Ising model

part 2 definitions and main theorems

Advanced Topics in Statistical Physics
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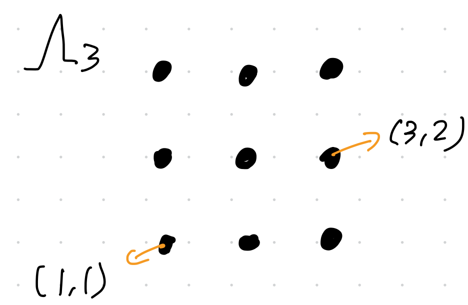
§ lattices

$L = 1, 2, \cdots$

set of sites

$L \times L$ square lattice $\quad \Lambda_L = \{1, 2, \dots, L\}^2$ (1)



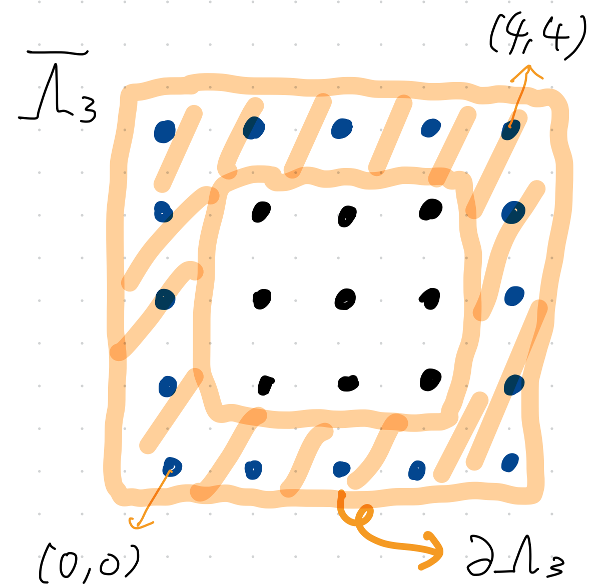
$\Lambda_L \ni u = (u_1, u_2)$ (2)

site

$u_1, u_2 \in \{1, \cdots, L\}$ (3)

larger lattice

$\bar{\Lambda}_L = \{0, 1, 2, \cdots, L, L+1\}^2$ (4)



boundary

$\partial\Lambda_L = \bar{\Lambda}_L \setminus \Lambda_L$ (5)

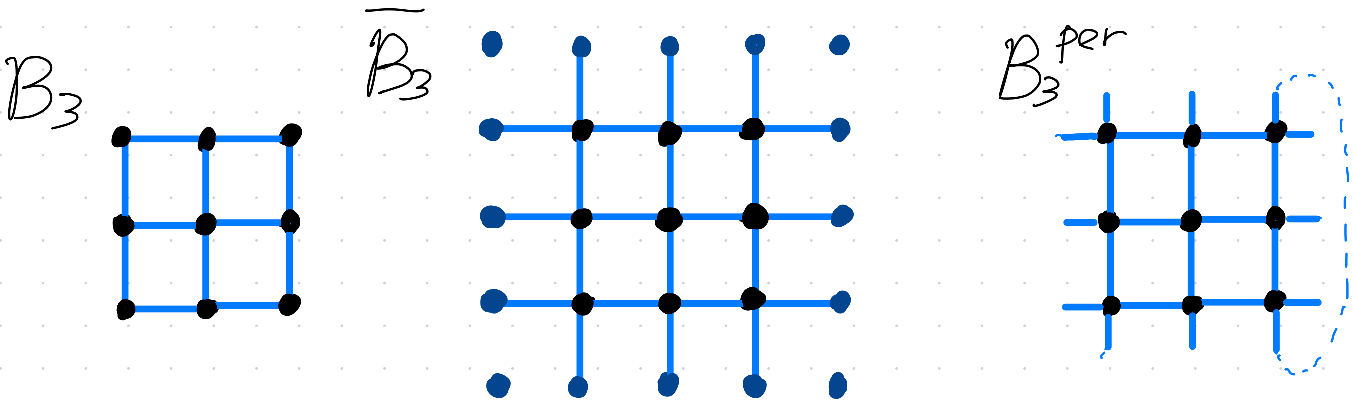
§ sets of bonds

bond $\{u,v\} = \{v,u\}$ unordered pair of sites

$$\mathcal{B}_L = \{\{u,v\} \mid u,v \in \Lambda_L,\ |u-v| = 1\} \quad (1)$$

$$\bar{\mathcal{B}}_L = \{\{u,v\} \mid u,v \in \bar{\Lambda}_L, \text{ but not } u,v \in \partial\Lambda_L,\ |u-v| = 1\} \quad (2)$$

$$\mathcal{B}_L^{per} = \mathcal{B}_L \cup \{\{(1,u_2),(L,u_2)\} \mid u_2 = 1,\dots,L\} \cup \{\{(u_1,1),(u_1,L)\} \mid u_1 = 1,\dots,L\} \quad (3)$$



§ Hamiltonians under three boundary conditions

Spin configuration $\sigma = (\sigma_u)_{u \in \Lambda_L}$ (1) $\qquad \sigma_u = \pm 1$ (2)

$\mathcal{S}_L$: set of all possible $\sigma \longrightarrow 2^{L^2}$ configurations

$H^{BC}_{L,h}(\sigma)$: Hamiltonian with magnetic field $h \in \mathbb{R}$ and boundary condition BC $=$ free, per, $+$

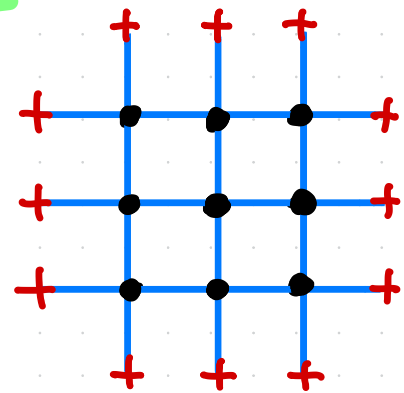
free b.c.

ferromagnetic interaction

$$H^{free}_{L,h}(\sigma) = -\sum_{\{u,v\} \in B_L} \sigma_u \sigma_v - h \sum_{u \in \Lambda_L} \sigma_u \quad (3)$$

periodic b.c

$$H^{per}_{L,h}(\sigma) = -\sum_{\{u,v\} \in B^{per}_L} \sigma_u \sigma_v - h \sum_{u \in \Lambda_L} \sigma_u \quad (4)$$



plus b.c.

$$H^{+}_{L,h}(\sigma) = -\sum_{\{u,v\} \in \overline{B}_L} \sigma_u \sigma_v - h \sum_{u \in \Lambda_L} \sigma_u \quad (5)$$

with $\sigma_v = 1$ for $v \in \partial\Lambda_L$

§ canonical distribution at inverse temperature $\beta > 0$

$$\beta = \frac{1}{k_B T}$$

partition function

$$Z_L^{BC}(\beta, h) = \sum_{\sigma \in \mathcal{S}_L} e^{-\beta H_{L,h}^{BC}(\sigma)} \qquad (1)$$

free energy density

$$f_L^{BC}(\beta, h) = -\frac{1}{\beta L^2} \log Z_L^{BC}(\beta, h) \qquad (2)$$

expectation value of a function $G(\sigma)$

$$\langle G(\sigma) \rangle_{L,\beta,h}^{BC} = \frac{1}{Z_L^{BC}(\beta, h)} \sum_{\sigma \in \mathcal{S}_L} G(\sigma)\, e^{-\beta H_{L,h}^{BC}(\sigma)} \qquad (3)$$

BC = free, per, +

expression of the magnetization density for finite L

$$-\sum_{u\in\Lambda_L}\sigma_u$$

$$\frac{\partial}{\partial h}Z_L^{BC}(\beta,h) = -\beta\sum_{\sigma\in\mathcal{S}_L}\boxed{\left(\frac{\partial}{\partial h}H_{L,h}^{BC}(\sigma)\right)}\, e^{-\beta H_{L,h}^{BC}(\sigma)} \qquad (1)$$

$$\left\langle\sum_{u\in\Lambda_L}\sigma_u\right\rangle_{L,\beta,h}^{BC}$$

$$\frac{\partial}{\partial h}\log Z_L^{BC}(\beta,h) = \frac{\frac{\partial}{\partial h}Z_L^{BC}(\beta,h)}{Z_L^{BC}(\beta,h)} = \boxed{\frac{\beta}{Z_L^{BC}(\beta,h)}\sum_{\sigma\in\mathcal{S}_L}\left(\sum_{u\in\Lambda_L}\sigma_u\right)e^{-\beta H_{L,h}^{BC}(\sigma)}} \qquad (2)$$

$$\frac{\partial}{\partial h}f_L^{BC}(\beta,h) = -\left\langle\frac{1}{L^2}\sum_{u\in\Lambda_L}\sigma_u\right\rangle_{L,\beta,h}^{BC} \qquad (3)$$

the expectation value of the magnetization density

§ main Theorems

theorem 3 for any $\beta > 0$ and $h \in \mathbb{R}$, the limit

$$f(\beta, h) = \lim_{L \uparrow \infty} f_L^{BC}(\beta, h) \quad (1)$$

exsits, and is independent of BC = free, per, +

$f(\beta, h)$ is concave in h and satisfies $f(\beta, h) = f(\beta, -h)$

$f(\beta, h)$ represents (idealized) thermodynamic property of the system

$L \uparrow \infty$ limit $=$ thermodynamic limit

magnetization density

$$m(\beta, h) = -\frac{\partial}{\partial h} f(\beta, h) \quad (2) \quad \text{(when } f(\beta,h) \text{ is differentiable in } h\text{)}$$

see part 1 p2-(3)

if $f(\beta,h)$ is differentiable in h at $h=0$

$$m(\beta,0) = -\frac{\partial}{\partial h} f(\beta,0) = 0 \quad (1)$$ because $f(\beta,h)=f(\beta,-h)$

spontaneous magnetization

$$m_s(\beta) = -\boxed{\lim_{h\downarrow 0}\frac{f(\beta,h)-f(\beta,0)}{h}} \quad (2)$$

$= \frac{\partial}{\partial h_+} f(\beta,0)$ right-derivative.

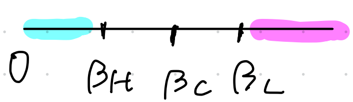
the limit exists because $f(\beta,h)$ is concave in h

theorem 4 there is $\beta_H \in (0,\infty)$ s.t. $m_s(\beta)=0$ for any $\beta\in(0,\beta_H)$

theorem 5 there is $\beta_L \in (0,\infty)$ s.t. $m_s(\beta)>0$ for any $\beta\in(\beta_L,\infty)$
we also have $m_s(\beta)\to 1$ as $\beta\uparrow\infty$

$\hookrightarrow$ $f(\beta,h)$ is NOT differentiable in h at $h=0$!

there is a phase transition between β_H and β_c



§ remarks

the proof is difficult

another characterization of the spontaneous magnetization

$$m_s(\beta) = \lim_{h \downarrow 0} \lim_{L \uparrow \infty} \left\langle \frac{1}{L^2} \sum_{u \in \Lambda_L} \sigma_u \right\rangle^{BC}_{L,\beta,h} \quad (1) \quad \text{for } BC = \text{free, per, } +$$

for BC = free, per, the symmetry $H^{BC}_{L,0}(\sigma) = H^{BC}_{L,0}(-\sigma)$ (2)

implies $\left\langle \frac{1}{L^2} \sum_{u \in \Lambda_L} \sigma_u \right\rangle^{BC}_{L,\beta,0} = 0$ (3) for any β and L

thus

$$\lim_{L \uparrow \infty} \lim_{h \downarrow 0} \left\langle \frac{1}{L^2} \sum_{u \in \Lambda_L} \sigma_u \right\rangle^{BC}_{L,\beta,h} = 0 \quad (4) \quad \text{for any } \beta$$

the order of the limits cannot be exchanged!

(1) $\Rightarrow$ the symmetry (2), (3) is broken by an infinitessimally small h

Spontaneous symmetry breaking (SSB)

more refined results for models in d-dimensions with any $d \geq 2$

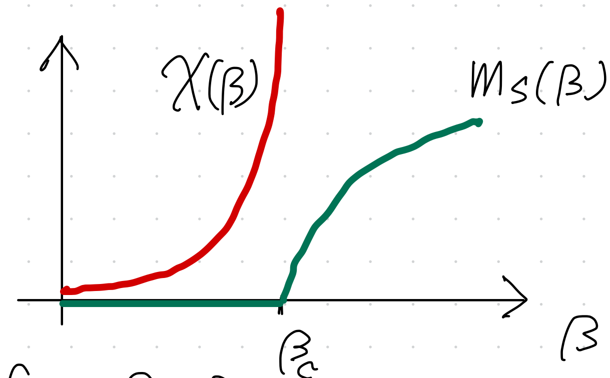
the proofs are difficult

$m(\beta, h) = -\frac{\partial}{\partial h} f(\beta, h)$ (1) exists and is continuous in h if $h \neq 0$

$\exists\, \beta_c \in (0, \infty)$ which depends only on d

$\beta_c = \frac{1}{2}\log(\sqrt{2}+1) \simeq 0.44$ in $d=2$

$$m_s(\beta) \begin{cases} = 0 & \beta \leq \beta_c \\ > 0 & \beta > \beta_c \end{cases} \quad (2)$$



$m_s(\beta) \downarrow 0$ as $\beta \downarrow \beta_c$ (3)

susceptibility

$\chi(\beta) = \frac{\partial}{\partial h} m(\beta, h)\Big|_{h=0} < \infty$ for $\beta < \beta_c$ (4)

$\chi(\beta) \uparrow \infty$ as $\beta \uparrow \beta_c$ (5)