Proof of the existence of a phase transition in the two-dimensional Ising model

part 1 thermodynamics and phase transition in a uniaxial ferromagnet

Advanced Topics in Statistical Physics
by Hal Tasaki

§ thermodynamic description of a uni-axial ferromagnet

See, e.g., H. Tasaki and G. Paquette, Thermodynamics – A modern approach (Oxford University Press, 2025?)

T : temperature, h : external magnetic field, N : amount of substance

intensive extensive

free energy (complete thermodynamic function)

$F[T, h; N]$ sometimes denoted as $G[T, h; N]$

Legendre transformation $F[T; M, N]$

free energy density $f(T,h) = \frac{1}{N} F[T; h, N]$ (1)

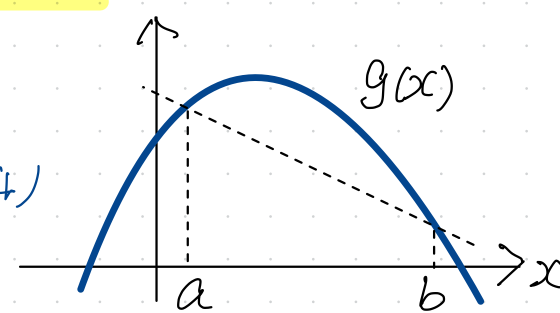
$f(T,h)$ is concave in h and satisfies $f(T,h) = f(T,-h)$ (2)

magnetization density $m(T,h) = -\frac{\partial}{\partial h} f(T,h)$ (3)

a function $g(x)$ is concave in x if

$g(\lambda a + (1-\lambda) b) \geq \lambda g(a) + (1-\lambda) g(b)$ (4)

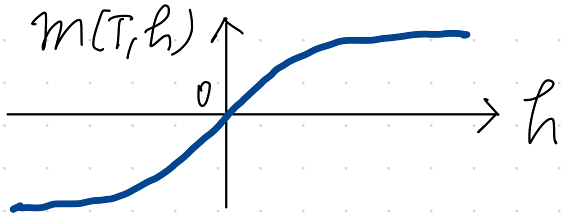
for any a, b, and $\lambda \in [0,1]$

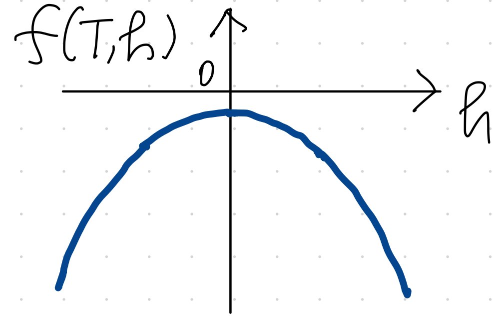


§ ferromagnetic phase transition

T_c : Curie temperature (or the transition temperature)

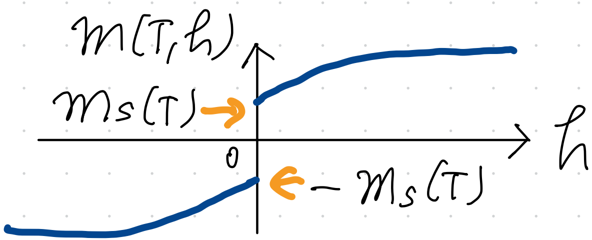
① high-temperature region $T > T_c$

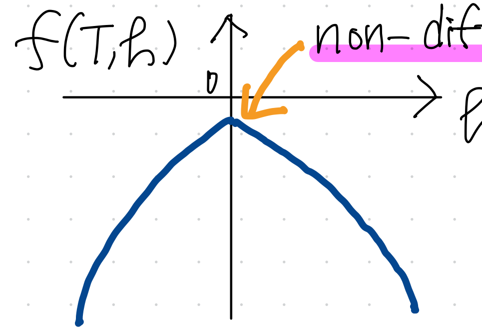




$m(T,0) = 0$ no magnetization if no magnetic field

② low-temperature region $T < T_c$





non-differentiable!!

$m(T,h) \xrightarrow{h \downarrow 0} m_s(T)$, $m(T,h) \xrightarrow{h \uparrow 0} -m_s(T)$

nonzero magnetization at zero magnetic field!!

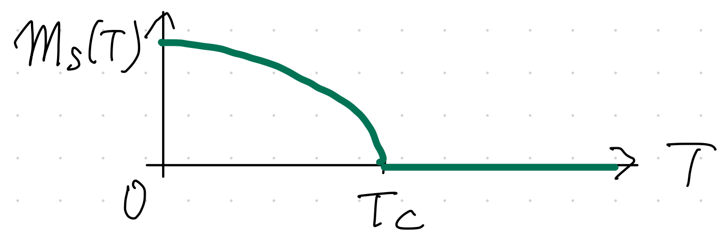
Spontaneous magnetization $m_s(T)$

$$m_s(T) = \lim_{h \downarrow 0} m(T,h) \overset{(*)}{=} -\boxed{\frac{\partial}{\partial h_+} f(T,0)} = \lim_{h\downarrow 0} \frac{f(T,h)-f(T,0)}{h} \quad (1)$$

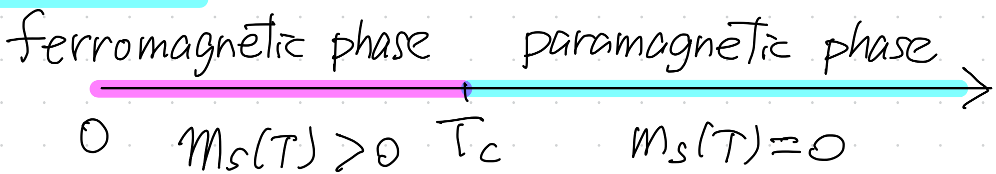
right derivative

(the existence of the limits and the equality (*) follow from the concavity of $f(T,h)$ see, e.g., Tasaki–Paquette)

$$m_s(T) \begin{cases} = 0, & T \geq T_c \\ > 0, & T < T_c \end{cases} \quad (2)$$



phase diagram ($h=0$)



spontaneous magnetization $m_s(T)$ is an order parameter that characterizes the ferromagnetic phase transition

§ Statistical mechanics and phase transitions

$$f(T,h) \text{ is } \begin{cases} \text{differentiable in } h \text{ for } T \geq T_c \\ \text{non-differentiable in } h \text{ at } h=0 \text{ for } T < T_c \end{cases}$$

can statistical mechanics describe singular behaviors as non-differentiability??

recall

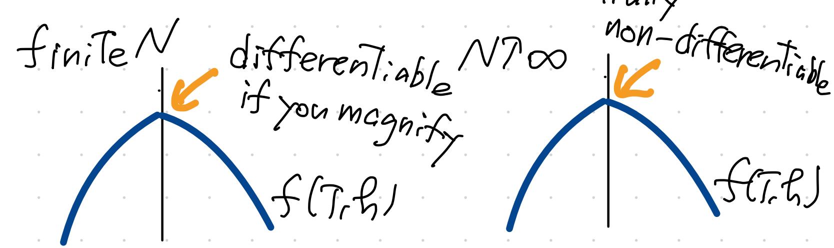
$$f(T,h) = -\frac{kT}{N} \log \underbrace{\sum_{j=1}^{\Omega}}_{\text{a finite sum}} \underbrace{e^{-\frac{1}{kT}(E_j - hM_j)}}_{\text{a nice function of } h} \qquad (1)$$

$f(T,h)$ can never be singular! as long as N is finite

we need to consider a proper thermodynamic limit $N \nearrow \infty$ → idealization

↓

universal thermodynamic behavior
(including phase transitions)



§ The Lenz-Ising model (usually called the Ising model)

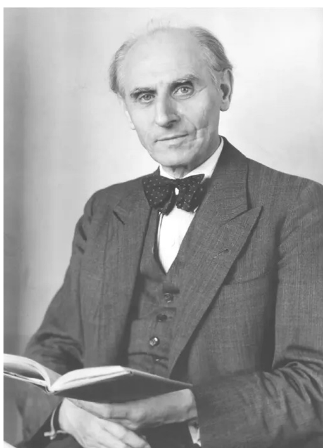
an over-simplified model of uni-axial ferromagnets

spin $\sigma_u = \pm 1 \rightarrow$ only up or down   simple interaction $\sim \sigma_u \sigma_{u'}$

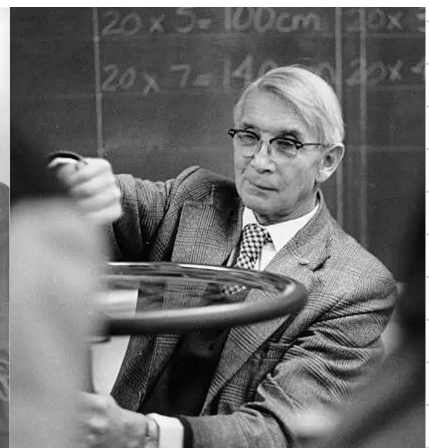
1920 Lenz proposed the model

1925 Ising solved the 1-dimensional model and found no phase transition

1944 Onsager solved the 2 dim. model with $h=0$ exactly and found a phase transition



Wilhelm Lenz (1888-1957)



Ernst Ising (1900-1988)

Lars Onsager (1903-1976)

the main topic of this mini-course

1936 Peierls argued that the model in 2 or higher dim. should have a phase transition

1964 Griffiths
1965 Dobrushin
proved the existence of phase transition in 2 and higher dimensions

Photo by G. Grimmett

Harry Kesten (1931–2019)

Rudolf Peierls (1907–1995)

Roland Dobrushin (1929–1995)

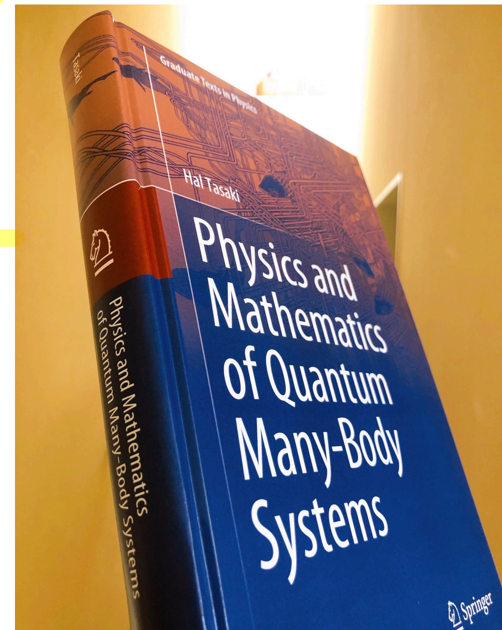
Robert Griffiths (1937–)

Photo from the web page of Carnegie Mellon Univeristy

so many important / beautiful works ...

The Lenz-Ising model allows us to study universal aspects of physical systems with many mutually interacting degrees of freedom!

In our pursuit to advance fundamental science, with the manner of thinking described above, we are led to study classes of systems defined by characteristic types of universal phenomena, rather than actual individual systems. Let us call such a class of systems a "universality class."[4] Within a given universality class, we will have not only actual physical systems and faithful theoretical models describing them (which usually have intractable details), but also some idealized theoretical models that appear to be easier to treat. It should be stressed, however, that such idealized models are not simply "made up" to exhibit the desired properties (for some obvious reasons). Rather, they are nontrivial systems that capture only the essence of the phenomena that we wish to understand.[5] By studying such idealized models, we are able to directly confront the problem of elucidating the essential behavior of interest. Perhaps the best example of such an idealized model is the (classical) Ising model. Although the Ising model is now recognized as a model of a ferromagnet, it is too simple to be a faithful model of any actual magnetic system. Nevertheless, we can learn from the Ising model extremely rich essence of phase transitions and critical phenomena associated with the breakdown of $\mathbb{Z}_2$ symmetry, exhibited by various physical systems, including uniaxial ferromagnets and some quantum field theories. It should be pointed out that, despite its relative simplicity, the Ising model is certainly not easy to solve. However, because with this model, one need not treat some of the very complicated problems involved with more realistic models, such as the overlap of electron orbits that determines the exchange interaction and the ultraviolet divergence that must be removed to realize a well-defined field theory, the core problem that we wish to address — that of describing the collective behavior of infinitely many interacting degrees of freedom — is laid bare. This problem is indeed central to understanding the large-scale behavior of a truly vast range of physical systems.



Hal Tasaki "Physics and Mathematics of Quantum Many-Body Systems", (Springer, 2020), page 3

§ plan

part 1 thermodynamics and phase transition in a uniaxial ferromagnet

part 2 definitions and main theorems

part 3 the existence of the infinite volume limit

part 4 high-temperature region

part 5 low-temperature region